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A GENERALLY COVARIANT FIELD EQUATION FOR GRAVITATION AND ELECTROMAGNETISM

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A generally covariant field equation is developed for gravitation and electromagnetism by considering the metric vector q^μ in curvilinear, non-Euclidean spacetime. The field equation is

$$R^\mu - \frac{1}{2}Rq^\mu = kT^\mu,$$

where T^μ is the canonical energy-momentum four-vector, k the Einstein constant, R^μ the curvature four-vector, and R the Riemann scalar curvature. It is shown that this equation can be written as

$$T^\mu = \alpha q^\mu,$$

where α is a coefficient defined in terms of R, k, and the scale factors of the curvilinear coordinate system. Gravitation is described through the Einstein field equation, which is recovered by multiplying both sides by q^μ. Generally covariant electromagnetism is described by multiplying the foregoing on both sides by the wedge $\wedge q^\nu$. Therefore, gravitation is described by symmetric metric $q^\mu q^\nu$ and electromagnetism by the anti-symmetric defined by the wedge product $q^\mu \wedge q^\nu$.

Key words: generally covariant field equation for gravitation and electromagnetism, $O(3)$ electrodynamics $\mathbf{B}^{(3)}$ field.

1. INTRODUCTION

The principle of general relativity states that every theory of physics should be generally covariant, i.e., retain its mathematical form under

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the general coordinate transformation in non-Euclidean spacetime defined by any well-defined set of curvilinear coordinates [1]. This is a well known and accepted principle [2], so a unified field theory should also be generally covariant. At present, however, only one out of the four known fields of nature, gravitational, electromagnetic, weak and strong, is described by a generally covariant field equation to wit, the Einstein field equation of gravitation

$$R^{\mu\nu(S)} - \frac{1}{2}Rq^{\mu\nu(S)} = kT^{\mu\nu(S)}. \tag{1}$$

Here $q^{\mu\nu(S)}$ is the symmetric metric tensor, $R^{\mu\nu(S)}$ the symmetric Ricci tensor defined in Riemann geometry, R the scalar curvature, k the Einstein constant, and $T^{\mu\nu(S)}$ the symmetric canonical energy-momentum tensor.

In this Letter, a generally covariant field equation for gravitation and electromagnetism is inferred through fundamental geometry: In non-Euclidean spacetime the existence of a symmetric metric tensor $q^{\mu\nu(S)}$ implies the existence of an anti-symmetric metric tensor $q^{\mu\nu(A)}$. The former is defined by the line element ds^2 formed from the square of the arc length and the latter by the area element dA. In differential geometry, the one-form ds^2 is dual to the two-form dA. The symmetric metric tensor is defined by the symmetric tensor product of two metric four-vectors:

$$q^{\mu\nu(S)} = q^{\mu}q^{\nu} \tag{2}$$

and the antisymmetric metric tensor by the wedge product:

$$q^{\mu\nu(A)} = q^{\mu} \wedge q^{\nu}, \tag{3}$$

where the metric four-vector is

$$q^{\mu} = (h^0, h^1, h^2, h^3). \tag{4}$$

Here h^i are the scale factors of the general covariant curvilinear coordinate system defining non-Euclidean spacetime. The scale factors can be real or complex numbers or matrices in Clifford algebra. Both types of metric tensor are therefore defined by the metric vector q^{μ}. From this result of geometry, it is inferred that if gravitation is identified through $q^{\mu\nu(S)}$, through the well-known Eq. (1), then electromagnetism is identified through $q^{\mu\nu(A)}$. This inference is developed in Sec. 3 into a generally covariant field equation of gravitation and electromagnetism, an equation written in terms of the metric four-vector q^{μ}, which is at the root of both gravitation and electromagnetism. The following section defines the fundamental geometrical concepts needed for the field equation inferred in Sec. 3.

2. FUNDAMENTAL GEOMETRICAL CONCEPTS

Restrict attention initially to three non-Euclidean space dimensions. The set of curvilinear coordinates is defined as (u_1, u_2, u_3), where the functions are single valued and continuously differentiable, and where there is a one-to-one relation between (u_1, u_2, u_3) and the Cartesian coordinates. The position vector is $\mathbf{r}(u_1, u_2, u_3)$ and the arc length is the modulus of the infinitesimal displacement vector:

$$ds = |d\mathbf{r}| = \left| \frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right| . \tag{5}$$

The metric coefficients are $\partial \mathbf{r}/\partial u^i$ and the scale factors are

$$h_i = \left| \frac{\partial \mathbf{r}}{\partial u_i} \right| . \tag{6}$$

The unit vectors are

$$\mathbf{e}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial u_1} \tag{7}$$

and obey the $O(3)$ symmetry cyclic relations

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \tag{8}$$

where $O(3)$ is the rotation group of three dimensional space [3-8]. The curvilinear coordinates are orthogonal if

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}_2 \cdot \mathbf{e}_3 = 0, \quad \mathbf{e}_3 \cdot \mathbf{e}_1 = 0. \tag{9}$$

The symmetric metric tensor is then defined through the line element, a one-form of differential geometry:

$$\omega_1 = ds^2 = q^{ij(S)} du_i \, du_j, \tag{10}$$

and the anti-symmetric metric tensor through the area element, a two-form of differential geometry:

$$\omega_2 = dA = -\frac{1}{2} q^{ij(A)} du_i \wedge du_j, \tag{11}$$

These results generalize as follows to the four dimensions of any non-Euclidean spacetime:

$$\omega_1 = ds^2 = q^{\mu\nu(S)} du_\mu du_\nu, \tag{12}$$

$$\omega_2 = {}^*\omega_1 = dA = -\frac{1}{2}q^{\mu\nu(A)}du_\mu \wedge du_\nu. \qquad (13)$$

In differential geometry, the element du_σ is dual to the wedge product $du_\mu \wedge du_\nu$.

The symmetric metric tensor is

$$q^{\mu\nu(S)} = \begin{bmatrix} h_0^2 & h_0h_1 & h_0h_2 & h_0h_3 \\ h_1h_0 & h_1^2 & h_1h_2 & h_1h_3 \\ h_2h_0 & h_2h_1 & h_2^2 & h_2h_3 \\ h_3h_0 & h_3h_1 & h_3h_2 & h_3^2 \end{bmatrix}, \qquad (14)$$

and the anti-symmetric metric tensor reads

$$q^{\mu\nu(A)} = \begin{bmatrix} 0 & -h_0h_1 & -h_0h_2 & -h_0h_3 \\ h_1h_0 & 0 & -h_1h_2 & h_1h_3 \\ h_2h_0 & h_2h_1 & 0 & -h_2h_3 \\ h_3h_0 & -h_3h_1 & h_3h_2 & 0 \end{bmatrix}. \qquad (15)$$

3. THE GENERALLY COVARIANT FIELD EQUATION

It has been shown that both the symmetric and anti-symmetric metric can be built up of individual metric four-vectors q^μ in any non-Euclidean spacetime, including the Riemannian spacetime used in Eq. (1). It can therefore be inferred that the Einstein field equation (1) can be built up from the generally covariant field equation:

$$R^\mu - \frac{1}{2}Rq^\mu = kT^\mu. \qquad (16)$$

Equation (1) is recovered from Eq. (16) by multiplying both sides of the latter by the metric four-vector q^ν. We may therefore define the familiar symmetric tensors appearing in the Einstein field equation of gravitation as follows:

$$R^{\mu\nu(S)} = R^\mu q^\nu, \qquad (17)$$

$$q^{\mu\nu(S)} = q^\mu q^\nu, \qquad (18)$$

$$T^{\mu\nu(S)} = T^\mu q^\nu \qquad (19)$$

in terms of the more fundamental four vectors R^μ, q^μ, and T^μ.

Equation (16) gives the generally covariant form of Newton's second law:

$$f^\mu = \frac{\partial T^\mu}{\partial \tau}, \tag{20}$$

where f^μ is a force four-vector and τ the proper time. Newton's third law and Noether's theorem (conservation of energy-momentum) are expressed through the invariant

$$T^\mu T_\mu = \text{constant}, \tag{21}$$

and Newton's Law of universal gravitation in generally covariant form is given, from Eq. (16), by

$$f^\mu = \frac{1}{k}\frac{\partial G^\mu}{\partial \tau}, \quad G^\mu := R^\mu - \frac{1}{2}Rq^\mu. \tag{22}$$

The results of generally covariant gravitational theory are also given by Eq. (16) because it is the basis of Einstein's field equation (1).

We have argued that the metric four vector q^σ is dual to the wedge product $q^\mu \wedge q^\nu$. From this fundamental result in differential geometry [8], it follows that the Einstein field equation is dual to the following equation between two forms:

$$R^\mu \wedge q^\nu - \frac{1}{2}Rq^\mu \wedge q^\nu = kT^\mu \wedge q^\nu, \tag{23}$$

an equation which is derived by multiplying both sides of Eq. (16) by the wedge $\wedge q^\nu$ and in which appear the anti-symmetric Ricci tensor, anti-symmetric metric, and anti-symmetric energy-momentum tensor, respectively, defined as follows:

$$R^{\mu\nu(A)} = R^\mu \wedge q^\nu, \tag{24}$$

$$q^{\mu\nu(A)} = q^\mu \wedge q^\nu, \tag{25}$$

$$T^{\mu\nu(A)} = T^\mu \wedge q^\nu. \tag{26}$$

Define the anti-symmetric field tensor

$$G^{\mu\nu(A)} = G^{(0)}\left(R^{\mu\nu(A)} - \frac{1}{2}Rq^{\mu\nu(A)}\right). \tag{27}$$

The anti-symmetry of the tensor implies the following Jacobi identity of non-Euclidean spacetime [3-8]:

$$D_\rho G^{(A)}_{\mu\nu} + D_\mu G^{(A)}_{\nu\rho} + D_\nu G^{(A)}_{\rho\mu} := 0, \tag{28}$$

where D_μ are generally covariant four-derivatives. In Riemannian spacetime, they can be defined through Christoffel symbols that are anti-symmetric in their lower two indices. The Jacobi identity (28) can be rewritten as

$$D_\mu \widetilde{G}^{\mu\nu(A)} := 0, \tag{29}$$

where

$$\widetilde{G}^{\mu\nu(A)} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^{(A)}_{\rho\sigma} \tag{30}$$

is the dual of $G^{(A)}_{\rho\sigma}$ and orthogonal to $G^{(A)}_{\rho\sigma}$:

$$\widetilde{G}^{\mu\nu(A)} G^{(A)}_{\mu\nu} = 0. \tag{31}$$

Taking covariant derivatives, either side of Eq. (23) gives:

$$D_\mu G^{\mu\nu(A)} = G^{(0)} k D_\mu T^{\mu\nu(A)}. \tag{32}$$

Define the generally covariant four-vector

$$j^\nu = \mu_0 G^{(0)} k D_\mu T^{\mu\nu(A)}, \tag{33}$$

where μ_0 is the vacuum permeability, a fundamental constant [3-8]. Then Eq. (32) can be written as

$$D_\mu G^{\mu\nu(A)} = j^\nu / \mu_0. \tag{34}$$

We define Eqs. (29) and (34) as, respectively, the homogenous and inhomogeneous field equations of generally covariant electromagnetism.

The generally covariant electric and magnetic fields are defined as

$$\begin{aligned} cB^k &= -\frac{1}{2}e^{ijk} G^{ij(A)}, \quad E^k = -G^{0k(A)} \\ \tilde{E}^k &= -\frac{1}{2}e^{ijk} \widetilde{G}^{ij(A)}, \quad c\widetilde{B}^k = -G^{0k(A)}, \end{aligned} \tag{35}$$

and the generally covariant electromagnetic field tensors as

$$G^{\mu\nu(A)} = \begin{bmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -cB^3 & cB^2 \\ E^2 & cB^3 & 0 & -cB^1 \\ E^3 & -cB^2 & cB^1 & 0 \end{bmatrix}, \quad \widetilde{G}^{\mu\nu(A)} = \begin{bmatrix} 0 & -cB^1 & -cB^2 & -cB^3 \\ cB^1 & 0 & E^3 & -E^2 \\ cB^2 & -E^3 & 0 & E^1 \\ cB^3 & E^2 & -E^1 & 0 \end{bmatrix}. \tag{36}$$

where the proportionality coefficient is defined as

$$\alpha := \frac{R}{k}\left(\frac{1}{h^4} - \frac{1}{2}\right). \tag{44}$$

On the unit hypersphere

$$h_0^2 - h_1^2 - h_2^2 - h_3^2 = 1, \tag{45}$$

the proportionality simplifies to

$$\alpha = R/2k. \tag{46}$$

Using the definition (17) for the symmetric Ricci tensor and multiplying on both sides by q_ν, we also obtain the following useful expression for the curvature four-vector as a contraction of the symmetric Ricci tensor in Riemann geometry:

$$R^\mu = (1/h^2) q_\nu R^{\mu\nu(S)}. \tag{47}$$

Equation (43) shows that for both gravitation and electromagnetism the generally covariant energy momentum four-vector T^μ is proportional to the generally covariant metric four-vector q^μ through the metric dependent proportionality coefficient α. It is likely that such a result is also true for the weak and strong fields, because it is known that the electromagnetic field can be unified with the weak and strong field [3-8] and because both the weak and strong fields are gauge fields. It is likely therefore that Eq. (16) is a generally covariant field equation of classical grand unified field theory. This result is required by the Principle of General Relativity.

In the special case where the covariant derivatives of the Yang Mills field theory have $O(3)$ internal symmetry, with indices (1),(2), and (3), where ((1),(2),(3)) is the complex circular representation of space, Eqs. (29) and (34) become the field equations of $O(3)$ electrodynamics [3-7]. The latter has been extensively discussed in the literature and tested against experimental data from several sources, and can now be recognised as an example of Eq. (16), illustrating the advantages of Eq. (16) over the received view of electromagnetism. Therefore $O(3)$ electrodynamics is a theory of general relativity, whereas Maxwell-Heaviside electrodynamics is a theory of special relativity in Euclidean spacetime in which the field is an entity superimposed on the frame of reference, a Yang Mills gauge field theory with $U(1)$ internal gauge group symmetry. The several advantages of $O(3)$ electrodynamics over the received opinion have been discussed in the literature [3-7] and it can now be seen that these advantages stem from the fact that Eq. (16) gives a theory of generally covariant electromagnetism and also the well-known generally covariant theory of gravitation. Using Eq. (43) it can be seen that both gravitation and electromagnetism are defined by the metric vector q^{μ} within the proportionality coefficient α, both fields being essentially the frame of reference itself. We may now conclude that the non-Euclidean nature of spacetime gives rise to both gravitation and electromagnetism through Eq. (16).

A similar conclusion has been reached by Sachs [9] using Clifford algebra, but the important Sachs unification scheme is based on Clifford algebra and is considerably more complicated than Eq. (16), which is therefore preferred by Ockham's Razor - choose the simpler of two theories.

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