## SOME NOTES ON THE SOLENOIDAL BELTRAMI EQUATIONS
## (Barrett and Grimes, pp. 228 ff).

This is

$$\nabla \times \boldsymbol{B} = k\boldsymbol{B}$$
$$\nabla \bullet \boldsymbol{B} = 0$$

Note that this is satisfied by:

$$\boldsymbol{B}^{(1)} = \frac{B^{(0)}}{\sqrt{2}}\left(i\boldsymbol{i} + \boldsymbol{j}\right) e^{i(\omega t - \boldsymbol{\kappa} \bullet \boldsymbol{r})}$$

$$\boldsymbol{B}^{(2)} = \frac{B^{(0)}}{\sqrt{2}}\left(-i\boldsymbol{i} + \boldsymbol{j}\right) e^{-i(\omega t - \boldsymbol{\kappa} \bullet \boldsymbol{r})}$$

$$\boldsymbol{B}^{(3)} = B^{(0)}\boldsymbol{k}$$

Specifically:

$$\nabla \times \boldsymbol{B}^{(1)} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial X} & \frac{\partial}{\partial Y} & \frac{\partial}{\partial Z} \\ ie^{i\phi} & e^{i\phi} & 0 \end{bmatrix} \frac{B^{(0)}}{\sqrt{2}}$$

$$= \left(-\boldsymbol{i}\frac{\partial}{\partial Z}e^{i\phi} + i\boldsymbol{j}\frac{\partial}{\partial Z}e^{i\phi}\right)\frac{B^{(0)}}{\sqrt{2}}$$

$$= \kappa\left(i\boldsymbol{i} + \boldsymbol{j}\right)e^{i\phi}\,\frac{B^{(0)}}{\sqrt{2}}$$

$$= \kappa\,\boldsymbol{B}^{(1)}$$

$$\nabla \times \boldsymbol{B}^{(1)} = \kappa\,\boldsymbol{B}^{(1)}; \qquad \nabla \bullet \boldsymbol{B}^{(1)} = 0 \qquad (1)$$

Similarly:

$$\nabla \times \boldsymbol{B}^{(2)} = \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial X} & \dfrac{\partial}{\partial Y} & \dfrac{\partial}{\partial Z} \\ -ie^{-i\phi} & e^{-i\phi} & 0 \end{bmatrix} \frac{B^{(0)}}{\sqrt{2}}$$

$$= \left( -\boldsymbol{i} \frac{\partial}{\partial Z} e^{-i\phi} - i\boldsymbol{j} \frac{\partial}{\partial Z} e^{-i\phi} \right) \frac{B^{(0)}}{\sqrt{2}}$$

$$= \kappa \left( -i\boldsymbol{i} + \boldsymbol{j} \right) e^{-i\phi} \frac{B^{(0)}}{\sqrt{2}}$$

$$= \kappa \boldsymbol{B}^{(2)}$$

$$\nabla \times \boldsymbol{B}^{(2)} = \kappa \boldsymbol{B}^{(2)}; \qquad \nabla \cdot \boldsymbol{B}^{(2)} = 0 \qquad (2)$$

Similarly:

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}; \qquad \nabla \cdot \boldsymbol{B}^{(3)} = 0. \qquad (3)$$

## CONCLUSION

The field $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$, $\boldsymbol{B}^{(3)}$ are solenoidal and Beltrami, and it is clear that this does **not** mean that $\boldsymbol{B}^{(1)}$, $\boldsymbol{B}^{(2)}$ and $\boldsymbol{B}^{(3)}$ are everywhere constant.

## THE CORRECT INTERPRETATION OF $\nabla \times \boldsymbol{B}^{(3)}$ AND $\nabla \cdot \boldsymbol{B}^{(3)}$

This is:

$$\nabla \times \boldsymbol{B}^{(3)} = -ig\nabla \times \left(\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}\right)$$
$$\nabla \cdot \boldsymbol{B}^{(3)} = -ig\nabla \cdot \left(\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}\right)$$

which shows that $\boldsymbol{B}^{(3)}$ is not everywhere constant, because $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is not everywhere constant. Specifically:

1)

$$\begin{aligned}\nabla \cdot \boldsymbol{B}^{(3)} &= -ig\left[\boldsymbol{A}^{(2)} \cdot \left(\nabla \times \boldsymbol{A}^{(1)}\right) - \boldsymbol{A}^{(1)} \cdot \left(\nabla \times \boldsymbol{A}^{(2)}\right)\right] \\ &= -ig\left(\boldsymbol{A}^{(2)} \cdot \boldsymbol{B}^{(1)} - \boldsymbol{A}^{(1)} \cdot \boldsymbol{B}^{(2)}\right) \\ &= 0\end{aligned}$$

It is clear that $\boldsymbol{A}^{(2)} = \boldsymbol{A}^{(1)*}$ are defined as propagating fields, and so $\boldsymbol{B}^{(3)}$ is a propagating field that exists if and only if $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ exists, at a phase number $\omega t - \kappa \cdot \boldsymbol{Z}$.

2)

$$\begin{aligned}\nabla \times \boldsymbol{B}^{(3)} &= ig\left[\boldsymbol{A}^{(1)}\left(\nabla \cdot \boldsymbol{A}^{(2)}\right) - \left(\nabla \cdot \boldsymbol{A}^{(1)}\right)\boldsymbol{A}^{(2)} + \left(\boldsymbol{A}^{(2)} \cdot \nabla\right)\boldsymbol{A}^{(1)} - \left(\boldsymbol{A}^{(1)} \cdot \nabla\right)\boldsymbol{A}^{(2)}\right] \\ &= \boldsymbol{0}\end{aligned}$$

3)

$$\begin{aligned}\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} &= -ig\frac{\partial}{\partial t}\left(\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}\right) \\ &= -ig\left(\frac{\partial A^{(1)}}{\partial t} \times \boldsymbol{A}^{(2)} + \boldsymbol{A}^{(1)} \times \frac{\partial A^{(2)}}{\partial t}\right) \\ &= \boldsymbol{0}\end{aligned}$$