NON-ABELIAN FIELD THEORY APPLIED TO ELECTRODYNAMICS: DEVELOPMENT OF THE FIELD EQUATIONS

ABSTRACT

The theoretical and experimental limitations of conventional Maxwellian field theory are discussed in terms of contemporary gauge field theory. An O(3) internal gauge symmetry is used as the basis for novel, non-linear, field equations of electrodynamics. The Maxwell-Heaviside equations can be recovered as particular solutions, together with a novel equation of motion for a fundamental, topological, magnetic field which magnetizes matter in the inverse Faraday effect, and which is derived rigorously from the topological magnetic monopole. The Coulomb, Gauss, Ampère-Maxwell and Faraday laws are given in the higher symmetry form of electrodynamics implied by the inverse Faraday effect and by the existence of the various topological phases. In this form of the field equations of electrodynamics, the potential is no longer an arbitrary mathematical construct, but is physically meaningful. The novel topological magnetic field implied by the field equation can be observed directly in the topological phases and in magneto-optical effects. Some consequences for quantum electrodynamics are discussed.

INTRODUCTION

The Maxwell-Heaviside field equations are gauge field equations whose internal gauge symmetry is U(1) {1-3}. This choice of gauge group symmetry is based on the assumption that the non-linear part of the commutator of covariant derivatives that defines the gauge field tensor is always zero under all conditions. However, there are several ways of showing that this assumption does not hold in general. For example, the ordinary Stokes parameter, S_3, that defines circular polarization on the classical level {4, 5}, is proportional in the vacuum to the conjugate product or commutator of potentials $\boldsymbol{A} \times \boldsymbol{A}^*$, which in a circular basis ((1), (2), (3)) can be written as $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ {6-9}. This result is incompatible with the assumption $\boldsymbol{A} \times \boldsymbol{A}^* =? \boldsymbol{0}$ which has to be made so that the Maxwell-Heaviside equations are equations of a U(1) symmetry gauge field theory. A second example is the inverse Faraday effect {10}, which is magnetization due to the same conjugate product $\boldsymbol{A} \times \boldsymbol{A}^*$ of circularly polarized electromagnetic radiation. Magneto-optic effects, in general, rely on the existence of a non-zero conjugate product $\boldsymbol{A} \times \boldsymbol{A}^*$ which, in the vacuum, is proportional for plane waves to the conjugate product of electric fields $\boldsymbol{E} \times \boldsymbol{E}^* = c^2\boldsymbol{B} \times \boldsymbol{B}^*$. Here, c is the vacuum speed of light and $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. A third example is the existence of the topological phases {11-14} which can be shown to be due to an area integral over a non-zero commutator of potentials, equal to a line integral through a non-Abelian Stokes theorem.

The observation of a non-zero $\boldsymbol{A} \times \boldsymbol{A}^*$ in these effects is incompatible with the assumption $\boldsymbol{A} \times \boldsymbol{A}^* = \boldsymbol{0}$ made in deriving Maxwell-Heaviside electrodynamics from general gauge field theory.

These phenomena indicate that the potential on a classical level has a physical significance. The rules of gauge transformation cannot be applied because these randomize the observable $\boldsymbol{A} \times \boldsymbol{A}^*$, the reason being that in the U(1) electrodynamics, a random quantity is added to $\boldsymbol{A}$ under gauge transformation, and the complex conjugate of this random quantity is added to $\boldsymbol{A}^*$. The topological phase effects {15} are, furthermore, essentially Aharonov-Bohm effects, again suggesting that the potential has a physical, gauge covariant role incompatible with its meaning in Maxwell-Heaviside electrodynamics as a mathematical subsidiary. Barrett {16} has pointed out several other effects in which the classical potential is a physical observable, and has presented evidence for the existence of a topological magnetic monopole in electrodynamics.

In general gauge field theory, it is natural to address these fundamental self-inconsistencies with a different internal symmetry group, one which allows for the existence of a non-zero commutator of potentials through a choice of covariant derivatives {17}. Since $\boldsymbol{A} \times \boldsymbol{A}^*$ is the signature of circular polarization in the third Stokes parameter, it is natural to choose the symmetry group O(3), the rotation group. The choice of covariant derivatives defines the field tensor in Section 2 of this paper. Section 3 then derives the homogeneous and inhomogeneous field equations of O(3) electrodynamics. The homogeneous field equation is a Jacobi identity first derived by Feynman, {18} and both the homogeneous and inhomogeneous parts are identifiable with Yang-Mills theory {19}. It is well known that the latter set out to generalize electrodynamics in 1954, essentially through the introduction of the covariant derivative. Section 4 writes out the field equations in full and shows that they are homomorphic with the Barrett field equations, written in SU(2) symmetry {16}. They are respectively the Coulomb, Gauss, Ampère-Maxwell and Faraday laws written in an O(3) symmetry rather than the usual U(1) symmetry. The Gauss law, for example, allows for the existence of a topological magnetic monopole and the field equations in general allow for the existence of a topological magnetic field {20}, defined by a non-zero commutator $\boldsymbol{A} \times \boldsymbol{A}^*$. This topological magnetic field ($\boldsymbol{B}^{(3)}$) is therefore directly observed in the Stokes S_3 parameter, the inverse Faraday effect and other magneto-optical effects, and in the topological phases. The field equations are non-linear, but under certain conditions can be linearized to three field equations in indices (1), (2) and (3). The two linear field equations in (1) and (2) are Maxwell-Heaviside type equations, while that in index (3) governs $\boldsymbol{B}^{(3)}$, a constant of motion. Being a topological field, it does not give rise to Faraday induction, as observed empirically. Section 5 gives a rigorous derivation of the topological magnetic field $\boldsymbol{B}^{(3)}$ from the topological magnetic monopole, and their observation is discussed in the Sagnac effect and interferometry. Finally, a discussion is given of some of the consequences of these field equations in nonlinear optics, quantum electrodynamics and unified field theory.

THE FIELD TENSOR

Electrodynamics can be derived from gauge theory by using a closed loop in Minkowski spacetime {17}: a round trip with covariant derivatives. Such a procedure is valid for any internal gauge group symmetry, and electrodynamics can be derived in consequence for any internal gauge group. A general, multi-component, field vector ψ is acted upon by an operator which transports it around a closed loop using the theory of infinitesimal generators. The result of the trip around the closed loop is expressed as:

$$\psi' = e^{ix}e^{iy}e^{-ix}e^{iy}\psi. \tag{1}$$

The four exponentials are operators which can be expanded as a Taylor series. To second order, this takes the form:

$$e^{ix}e^{iy}e^{-ix}e^{iy} = 1-(xy-yx)+x^2+y^2+\ .\ .\ .\ .\ \ . \tag{2}$$

In order to apply this general theory to electrodynamics, the symbols x and y are defined by:

$$x = D_\mu \Delta x^\mu$$
$$y = D_\nu \Delta x^\nu \tag{2a}$$

where D_μ and D_ν are covariant derivatives and where x^μ and x^ν are four vectors in flat, Minkowski, space-time {17}. The covariant derivatives can be defined in any gauge group symmetry and can be expressed in the short-hand notation {17}:

$$D_\mu \equiv \partial_\mu - igA_\mu. \tag{3}$$

Using eqns. (1) and (2):

$$\psi' = \left(1 - \left[D_\mu, D_\nu\right]\Delta x^\mu \Delta x^\nu + D_\mu D^\mu \Delta x_\mu \Delta x^\mu + D_\nu D^\nu \Delta x_\nu \Delta x^\nu + \ .\ .\ .\ .\right). \tag{4}$$

To second order, the closed - loop journey gives a commutator containing covariant derivatives, i.e. xy - yx, and quadratic products such as x^2 and y^2. The field tensor of electrodynamics is given by the commutator, and can be expressed as:

$$G_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \tag{5}$$

in terms of the potentials A_μ. The field tensor is part of the commutator of covariant derivatives $[D_\mu, D_\nu]$, derivatives which obey the Jacobi identity for all group symmetries:

$$\sum\left[D_\rho, \left[D_\mu, D_\nu\right]\right] \equiv 0. \tag{6}$$

The Jacobi identity is the homogeneous field equation {17}, which is therefore also an identity. These are well known results of gauge theory, and electrodynamics is usually recovered by assuming that the internal gauge symmetry is U(1), giving the usual four-curl:

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

linear in potential.

This process is however, self-inconsistent as discussed in the introduction. It is more self-consistent if an O(3) symmetry is used for the internal gauge space. In the complex representation of space, with basis ((1), (2), (3)), that part of the field tensor quadratic in the potential is non-zero if the internal gauge space has O(3) symmetry {20}. The field component thus defined is the topological {21} magnetic field:

$$\boldsymbol{B}^{(3)*} \equiv -ig\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(3)} \tag{8}$$

where g is a proportionality constant {20}. This is a physical field component in the vacuum, giving rise to observable effects when field interacts with matter, for example, one electron. The inverse Faraday effect, for example, is due to a non-zero $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$, and therefore to magnetization by $\boldsymbol{B}^{(3)}$.

The O(3) field tensor written out in full is:

$$\begin{aligned} \boldsymbol{G}^{(1)*} &\equiv c\mu_0 \boldsymbol{H}^{(1)*} = c\nabla \times \boldsymbol{A}^{(1)*} - icg\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(3)} \\ \boldsymbol{G}^{(2)*} &\equiv c\mu_0 \boldsymbol{H}^{(2)*} = c\nabla \times \boldsymbol{A}^{(2)*} - icg\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \\ \boldsymbol{G}^{(3)*} &\equiv c\mu_0 \boldsymbol{H}^{(3)*} = c\nabla \times \boldsymbol{A}^{(3)*} - icg\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \end{aligned} \tag{9}$$

where c is the speed of light in vacuo, μ_0 is the vacuum permeability, and $\boldsymbol{H}$ is the magnetization/polarization vectors of the O(3) symmetry electrodynamics. There are several differences between the U(1) and O(3) field tensors, the main one being that the O(3) field tensor is quadratic in the potential. If $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)}$ is a plane wave in the radiation zone, the three physical magnetic field components are given by {20}:

$$\begin{aligned} \boldsymbol{B}^{(1)} &= \nabla \times \boldsymbol{A}^{(1)} = -ig\boldsymbol{A}^{(2)} \times \boldsymbol{A}^{(3)} \\ \boldsymbol{B}^{(2)} &= \nabla \times \boldsymbol{A}^{(2)} = -ig\boldsymbol{A}^{(3)} \times \boldsymbol{A}^{(1)} \\ \boldsymbol{B}^{(3)} &= \nabla \times \boldsymbol{A}^{(3)} = -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \end{aligned} \tag{10}$$

When $A^{(3)}$ is a constant, we recover eqn. (8). The proportionality factor g is definable in the vacuum through the duality {20}:

$$eA^{(0)} = \hbar\kappa \tag{11}$$

where $A^{(0)}$ is the scalar magnitude of $A^{(3)}$ and where κ is the wave-vector magnitude. The ratio of the Dirac constant $\hbar$ to the elementary charge e is the magnetic flux carried by the photon in the vacuum {20}. Special relativity {22} shows that two charges e co-moving at c do not interact and so there is no photon photon interaction in the vacuum due to $\hbar/e$.

When there is field matter interaction, the constant g changes its value so that {20}:

$$g \rightarrow g' = \frac{\mu_0}{\mu} g \tag{12}$$

where μ is the permeability of the substance with which the field interacts. The magnetization of the inverse Faraday effect, for example, is:

$$\boldsymbol{M}^{(3)}(\text{IFE}) = -\frac{i}{\mu_0} g' \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{13}$$

Therefore the inverse Faraday effect is described from first principles in O(3) electrodynamics, whereas in U(1) electrodynamics, it does not exist from the field equations without the phenomenological introduction of $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ in constitutive relations. Such a procedure violates the assumption made in deriving U(1) electrodynamics from gauge field theory, that terms such as $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ vanish.

When $\boldsymbol{A}^{(1)} = \boldsymbol{A}^{(2)}$ is a plane wave, the third Stokes parameter defining circular polarization on a classical level is {20}, in the vacuum:

$$S_3 \equiv -i\omega^2 \boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{14}$$

where ω is the angular frequency of the radiation. The linearization of the gauge field to eqn. (7) therefore removes the ability of the U(1) theory to describe the Stokes parameter, another severe self-inconsistency. In general, these self-inconsistencies appear whenever the optical phenomenon under consideration depends on features such as $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ which are quadratic in the potential. The fundamental reason is that the electromagnetic field is recovered from the underlying gauge field theory by discarding all terms non-linear in the potential. This procedure leads to the four-curl (7) and conventional, U(1), electrodynamics re-introduces non-linear terms when needed, as in the third Stokes parameter or in the inverse Faraday effect. Not only is this procedure self-inconsistent, but it also loses information, because the MacLaurin series (4) is truncated at the linear term.

THE FIELD EQUATIONS

The homogeneous field equation of electrodynamics, considered as a gauge field theory with O(3) internal symmetry, can be derived in the same way as the field tensor, by considering a closed path in Minkowski space-time. This theorem was first demonstrated by Feynman {17} in the context of high energy physics. The closed path gives the Jacobi identity (6). Using the relation between the generalized field tensor of electrodynamics {17} and the commutator of covariant derivatives:

$$G_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] \tag{15}$$

the Jacobi identity becomes the homogeneous field equation:

$$D_\mu \tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{16}$$

which is also a Jacobi identity for any gauge group symmetry. In O(3) symmetry, the covariant derivative is defined in terms of O(3) rotation generators, and the field tensor $\tilde{\boldsymbol{G}}^{\mu\nu}$ is a vector in the internal gauge space. If the gauge symmetry is U(1), eqn. (16) becomes the homogeneous Maxwell-Heaviside equation {17}.

The inhomogeneous field equation can be derived similarly and is:

$$D_\mu \boldsymbol{H}^{\mu\nu} = \boldsymbol{J}^\nu \tag{17}$$

where $\boldsymbol{H}^{\mu\nu}$ is a magnetization/polarization tensor and $\boldsymbol{J}^\nu$ is a four-current. In O(3) electrodynamics, both $\boldsymbol{H}^{\mu\nu}$ and $\boldsymbol{J}^\nu$ are vectors in the internal gauge space. Using the complex basis ((1), (2), (3)) {23} for this internal space, we can write each component in this space of the four-current:

$$J^{\mu(i)} \equiv \left(\rho^{(i)}, \frac{\boldsymbol{J}^{(i)}}{c}\right). \tag{18}$$

Eqns. (16) and (17) are more self-consistent than the Maxwell-Heaviside equations because the field tensor (Section (2)) defines $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ from first gauge field principles. Eqns. (16) and (17) have many other advantages as detailed in the literature {23-26}, in particular, they remove the fundamental inconsistencies discussed already. Therefore classical and quantum electrodynamics are described by solving eqns. (16) and (17) (or their quantum equivalents) in any given situation. A detailed example is given in this paper through the inverse Faraday effect. Most generally, the equations can be integrated numerically. They are similar to the Yang-Mills equations {17} in mathematical structure, and are therefore intrinsically non-linear. The Maxwell-Heaviside equations are linear differential equations. One of the major advantages of O(3) electrodynamics is that it defines the topological magnetic field $\boldsymbol{B}^{(3)}$ observed in the topological phases {27}. In general, the O(3) equations enrich electrodynamics philosophically as well as physically, bringing in to the subject new concepts, such as the physical and observable topological magnetic monopole {28} concomitant with $\boldsymbol{B}^{(3)}$, and the particle concomitant to the O(3) field: the instanton. The O(3) equations should not therefore be considered merely as mathematical generalizations of the Maxwell-Heaviside equations.

Particular solutions of eqns. (16) and (17) exist which show their relation to the Maxwell-Heaviside structure. Writing eqn. (16) as:

$$\left(\partial_\mu + g\boldsymbol{A}_\mu \times\right)\tilde{\boldsymbol{G}}^{\mu\nu} \equiv \boldsymbol{0} \tag{19}$$

a particular solution can be given:

$$\partial_\mu \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0} \tag{19a}$$

$$\boldsymbol{A}_\mu \times \tilde{\boldsymbol{G}}^{\mu\nu} = \boldsymbol{0} \tag{20}$$

which linearizes the O(3) homogeneous field equation. The components of eqn. (19) in the basis ((1), (2), (3)) are:

$$\partial_\mu \tilde{G}^{\mu\nu(1)} = \partial_\mu \tilde{G}^{\mu\nu(2)} = 0 \tag{21}$$

$$\partial_\mu \tilde{G}^{\mu\nu(3)} = 0. \tag{22}$$

Eqns. (21) and (22) are complex conjugate pairs of Maxwell-Heaviside equations while eqn. (22) is the equation of motion of $\boldsymbol{B}^{(3)}$. In vector notation, eqn. (22) is:

$$\frac{\partial \boldsymbol{B}^{(3)}}{\partial t} = \boldsymbol{0}, \tag{23}$$

which shows that $\boldsymbol{B}^{(3)}$ is a constant of motion if $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)*}$ is a transverse plane wave in vacuo {23-26}. The topological magnetic field $\boldsymbol{B}^{(3)}$ does not behave like a static magnetic field or a plane wave, and does not give rise to Faraday induction. It is observed in the third Stokes parameter, in the inverse Faraday effect and in the topological phases, a deduction which follows from the hypothesis that electrodynamics be an O(3) symmetry gauge field theory. It can be shown {26} that eqn. (20) self-consistently gives rise to the B Cyclic theorem {23-26}:

$$\boldsymbol{B}^{(1)} \times \boldsymbol{B}^{(2)} = iB^{(0)}\boldsymbol{B}^{(3)*} \quad \text{et cyclicum} \tag{24}$$

which in the vacuum is eqn. (8), a result which is consistent with the fact that the field tensor has O(3) gauge field symmetry (Section 2). Similarly, the inhomogeneous equation (17) in vacuo can gives the particular solution:

$$\partial_\mu \boldsymbol{G}^{\mu\nu} = \boldsymbol{0} \tag{25}$$

$$\boldsymbol{J}^\nu = \frac{g}{\mu_0 c^2} \boldsymbol{A}_\mu \times \boldsymbol{G}^{\mu\nu}, \tag{26}$$

showing that the $\boldsymbol{B}^{(3)}$ field is irrotational in this case:

$$\nabla \times \boldsymbol{B}^{(3)} = \boldsymbol{0}. \tag{27}$$

Eqn. (26) of this solution self-consistently gives rise {26} to the energy generated by $\boldsymbol{B}^{(3)}$:

$$En = \frac{1}{\mu_0} \int \boldsymbol{B}^{(3)} \cdot \boldsymbol{B}^{(3)} dV \tag{28}$$

where V is the volume of radiation. It is well known that the B Cyclic theorem {23-26} is a Lorentz covariant angular momentum relation, showing that $\boldsymbol{B}^{(3)}$ is a fundamental spin of the electromagnetic field. This is another concept that does not exist in Maxwell-Heaviside electrodynamics because of its assumption that $\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)}$ is zero, a linearizing assumption.

STRUCTURE OF THE FIELD EQUATIONS

Eqns. (16) and (17), when written out in the ((1), (2), (3)) basis, give four laws of electrodynamics, laws which are similar to the Gauss, Faraday, Coulomb and Ampère-Maxwell laws of U(1) electrodynamics, although they are considerably richer, philosophically and mathematically. Therefore, we will refer to them, for convenience, as the O(3) equivalents of these laws.

The O(3) Gauss Law

The existence of the topological magnetic monopole is demonstrated through the O(3) Gauss law:

$$\nabla\cdot\boldsymbol{B}^{(1)*} \equiv ig\left(\boldsymbol{A}^{(2)}\cdot\boldsymbol{B}^{(3)} - \boldsymbol{B}^{(2)}\cdot\boldsymbol{A}^{(3)}\right) \quad \text{et cyclicum.} \tag{29}$$

In the particular case where $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)}$ is a plane wave and where $\boldsymbol{B}^{(3)}$ is solenoidal and irrotational, the O(3) Gauss law becomes:

$$\nabla\cdot\boldsymbol{B}^{(i)} = 0. \tag{30}$$

In general, however, the law consists of three simultaneous partial differential equations, eqns. (29), which must be solved numerically for the quantities appearing in them.

The O(3) Faraday Induction Law

The Faraday induction law becomes, under all conditions:

$$\nabla\times\boldsymbol{E}^{(1)*} + \frac{\partial\boldsymbol{B}^{(1)*}}{\partial t} \equiv -ig\left(cA_0^{(3)}\boldsymbol{B}^{(2)} - cA_0^{(2)}\boldsymbol{B}^{(3)} + \boldsymbol{A}^{(2)}\times\boldsymbol{E}^{(3)} - \boldsymbol{A}^{(3)}\times\boldsymbol{E}^{(2)}\right) \quad \text{et cyclicum} \tag{31}$$

and when $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)}$ is a plane wave, reduces to:

$$\nabla\times\boldsymbol{E}^{(i)*} + \frac{\partial\boldsymbol{B}^{(i)*}}{\partial t} = \boldsymbol{0}; \qquad i = 1,2 \tag{32a}$$

$$\frac{\partial\boldsymbol{B}^{(3)*}}{\partial t} = \boldsymbol{0}. \tag{32b}$$

Eqn. (32a) are complex conjugate Faraday laws and eqn. (32b) is the law for the constant of motion, $\boldsymbol{B}^{(3)}$.

The Ampère-Maxwell Law

The Ampère-Maxwell law in O(3) electrodynamics can be developed as:

$$\partial_\mu \boldsymbol{H}^{\mu\nu(1)*} = \boldsymbol{J}^{\nu(1)*} + igA_\mu^{(2)}\times\boldsymbol{H}^{\mu\nu(3)} \quad \text{et cyclicum} \tag{33}$$

and applies to field matter interaction. It can be illustrated with respect to the inverse Faraday effect, which as we have argued, is self-inconsistently described with U(1) electrodynamics. Using the constitutive relation:

$$H^{\mu\nu(3)} = \varepsilon G^{\mu\nu(3)} \tag{34}$$

it becomes possible to write eqn. (33) as:

$$\partial_\mu H^{\mu\nu(1)*} = J^{\nu(1)*} + \Delta J^{\nu(1)*} \quad \text{et cyclicum} \tag{35}$$

where the transverse current

$$\Delta J^{\nu(1)*} = ig\varepsilon A_\mu^{(2)} \times G^{\mu\nu(3)} \tag{36}$$

causes a signal in an induction coil due to the vacuum $B^{(3)}$ appearing in $G^{\mu\nu(3)}$. This is the inverse Faraday effect as observed empirically {23-26} and as described from first gauge field theoretical principles from eqn. (17). On the other hand, an explanation such as this is not possible in U(1) electrodynamics because of the self-inconsistent re-introduction of $A^{(1)} \times A^{(2)}$ phenomenologically {29} into linear field equations through non-linear constitutive relations. The description of the inverse Faraday effect through the eqns. (33) is developed in more detail elsewhere.

The Coulomb Law

Finally the Coulomb Law in O(3) electrodynamics can be developed as:

$$\nabla \cdot E^{(1)*} = \frac{1}{\varepsilon_0} \rho^{(1)*} + ig\left(A^{(2)} \cdot E^{(3)} - E^{(2)} \cdot A^{(3)}\right) \quad \text{et cyclicum} \tag{37}$$

and contains a topological charge in addition to the point charge of U(1) electrodynamics.

The laws of O(3) electrodynamics developed in this section are homomorphic with Barrett's equations {30}, which are developed in SU(2) symmetry. Barrett's equations are reproduced for convenience below:

$$\begin{aligned}
&\nabla \cdot B = -iq\left(A \cdot B - B \cdot A\right) \\
&\nabla \times E + \frac{\partial B}{\partial t} = -iq\left(\left[A_0, B\right] + A \times E - E \times A\right) \\
&\nabla \cdot E = J_0 - iq\left(A \cdot E - E \cdot A\right) \\
&\frac{\partial E}{\partial t} - \nabla \times B + J = iq\left(A \times B - B \times A - \left[A_0, E\right]\right)
\end{aligned} \tag{38}$$

where his q is equivalent to our g, and where he uses Gaussian units instead of our S.I. units. However, since SU(2) symmetry in gauge field theory does not support the Aharanov-Bohm effect, {17} the O(3) group is the only one possible in which to develop a higher symmetry form of electrodynamics. In so doing, we introduce into electrodynamics the highly developed concepts of non-Abelian gauge field theory in high energy physics {17}. Most generally, eqns. (16) and (17) can be solved numerically as coupled non-linear differential equations, provided a constitutive relation such as eqn. (34) is introduced. Such a relation is always needed on the classical level because there are four unknowns in eqns. (16) and (17), while there are only two simultaneous equations.

It is always possible to reduce the vacuum O(3) laws to equations which look like the vacuum U(1) laws by using particular solutions such as (19) and (25). These introduce cyclic relations such as (24), which are essentially angular momentum relations of the free electromagnetic field in vacuo, and extra energy terms of the free field, such as eqn. (28). In order to observe the effect of the field, however, field-matter interaction must always be present, and in the inverse Faraday effect, for example, the unabridged non-linear equation (33) is needed.

Electro-statics and Magneto-statics

There are no longitudinal source currents in eqns. (33) because the source current of circularly polarized radiation is necessarily transverse, the charge in the source goes around in a circle about the (3) axis. The source does not move forward along the (3) axis and there is therefore no polar source current in the (3) axis, the longitudinal axis. As the angular velocity of the charge approaches zero, the source stops radiating and we obtain O(3) electrostatics in which:

$$\boldsymbol{E} = E \times \boldsymbol{i} = \boldsymbol{E}^{(1)} = \boldsymbol{E}^{(2)} \tag{39}$$

so

$$\boldsymbol{E}^{(1)} \times \boldsymbol{E}^{(2)} = \boldsymbol{E} \times \boldsymbol{E} = \boldsymbol{0}. \tag{40}$$

There is no radiated $\boldsymbol{B}^{(3)}$ field in O(3) electro-statics, showing that the topological magnetic field is a radiated field, not a static magnetic field. Therefore there is no topological magnetic monopole in electro-statics and magneto-statics and the O(3) equations reduce to the empirical laws of electro-statics and magneto-statics: the Coulomb, Ampère and Gauss laws. This result is consistent with the fact that in electro-statics and magneto-statics, the particular solutions (19) and (25) always hold because the $\boldsymbol{B}^{(3)}$ field vanishes in the static limit. Since $\boldsymbol{B}^{(1)} = \boldsymbol{B}^{(2)}$ is real in the static limit, the particular solutions (19) and (25) reduce to the laws of electro-statics and magneto-statics as observed empirically.

This procedure illustrates the important conclusion that the topological field $B^{(3)}$ is a property of radiation, and that an O(3) symmetry electrodynamics consistently reduces to the laws of electro-statics and magneto-statics. In the next section, the topological magnetic field is related to the topological magnetic monopole.

THE $\boldsymbol{B}^{(3)}$ FIELD, TOPOLOGICAL MAGNETIC MONOPOLE AND PHASE

The non-Abelian addendum of phase to the Maxwell-Heaviside theory by Wu and Yang {31} implies the existence in electrodynamics of low energy topological magnetic monopoles. Furthermore, instantons appear in electrodynamics as minimum action solutions of the self-dual Yang-Mills equations related to eqns. (16) and (17). The link between $\boldsymbol{B}^{(3)}$ and the topological magnetic monopole is given most simply by considering the non-Abelian Stokes theorem {32}:

$$g_M = \frac{1}{V} \oint A_\mu dx^\mu = -i \frac{g}{V} \iint \left[A_\mu , A_\nu^* \right] d\sigma^{\mu\nu} \tag{41}$$

where the topological magnetic monopole is defined in terms of the Wu-Yang phase, the line integral on the left hand side of eqn. (41). In electrodynamics considered as a U(1) gauge field theory, such a theorem does not exist, because the commutator in U(1) theory is defined to be identically zero under all circumstances {17}. It follows that the line integral and phase vanish in U(1) gauge field theory applied to electrodynamics, and so vanish in the Maxwell-Heaviside theory. In consequence, there is no topological magnetic monopole or $B^{(3)}$ field in Maxwell-Heaviside electrodynamics. The line integral, or phase, is however observed empirically as the topological phase {33}, and such an observation clearly signals the limits of Maxwell-Heaviside electrodynamics. The mathematical structure of eqn. (41) furthermore implies the existence of the $B^{(3)}$ field through the empirical observation of the topological phase. Conversely, $B^{(3)}$ is responsible for the topological phase. The reason is that the commutator of potentials on the right hand side of eqn. (41) gives the $B^{(3)}$ field directly (Section 2). There is therefore a direct link between the topological magnetic monopole, observed in the topological phase, and $B^{(3)}$. On the simplest level the relation is:

$$g_M = \frac{1}{V}\iint B^{(3)} dAr = \frac{\Phi^{(3)}}{V} \tag{42}$$

where $\Phi^{(3)}$ is the magnetic flux (weber) due to $B^{(3)}$. For one photon this is $\hbar/e$. The topological phase is given by {30}:

$$\phi = -g\oint A_\mu dx^\mu \tag{43}$$

as first shown by Simon {34}. It is the result of a closed loop in Minkowski space-time using covariant derivatives. This is essentially the same procedure that gives rise to the non-Abelian field tensor (Section 2) and the non-Abelian field equations (Section 3). The observation of the well known topological phases {33} is conclusive evidence for the fact that electrodynamics is a non-Abelian gauge field theory.

It follows that the topological magnetic field $B^{(3)}$ is responsible for the topological phase effects as that observed by Tomita and Chiao {34} using a helically wound fiber. A more general theory of the link between the topological magnetic monopole and $B^{(3)}$ will be developed elsewhere, but it is already clear, from the simple considerations of this section, that the topological phase implies the existence of both in electrodynamics. It may also be argued that $B^{(3)}$ gives a straightforward topological explanation of the Sagnac effect {35}, which is essentially the Tomita Chiao effect for one turn of the fiber. It is well known that U(1) theory has considerable difficulty in explaining the Sagnac effect.

The non-Abelian Stokes theorem (41) is a direct consequence of eqn. (8), defining $B^{(3)}$ from the O(3) field tensor. This link may be demonstrated at the simplest level as follows. Start with eqn. (8) in the form:

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{44}$$

and use $B^{(0)} = \kappa A^{(0)}$. Multiply both sides of eqn. (44) by $Ar \equiv \pi R^2$ to obtain:

$$\begin{aligned}\pi R\kappa A^{(0)}\boldsymbol{k}\cdot R\boldsymbol{k} &= \pi\kappa A^{(0)}\boldsymbol{R}\cdot\boldsymbol{R} \\ &= \boldsymbol{B}^{(3)}\cdot Ar\,\boldsymbol{k}.\end{aligned} \tag{45}$$

Now integrate to obtain the non-Abelian Stokes theorem:

$$2\pi\kappa A^{(0)}\oint \boldsymbol{R}\cdot d\boldsymbol{R} = \iint \boldsymbol{B}^{(3)}\cdot d\boldsymbol{A}\boldsymbol{r}. \tag{46}$$

Finally let $R \equiv \kappa^{-1}$ and multiply both sides of eqn. (46) by $g = \kappa/A^{(0)}$ to define the phase:

$$\phi = 2\pi\oint \kappa\cdot d\boldsymbol{R} = \frac{\kappa}{A^{(0)}}\iint \boldsymbol{B}^{(3)}\cdot d\boldsymbol{A}\boldsymbol{r}. \tag{47}$$

The left hand sides defines the non-Abelian dynamical phase over a suitable closed loop {36}, the right hand side the non-Abelian topological phase, essentially the magnetic flux due to $B^{(3)}$. These phases are observables of optics and interferometry {36, 37}.

On the general level, the non-Abelian Stokes theorem defines the electromagnetic phase through:

$$\phi = i\oint D_\mu dx^\mu = i\iint \left[D_\mu , D_\nu \right] d\sigma^{\mu\nu} \tag{48}$$

and using eqn. (15) we find that:

$$\phi = \oint \left(\kappa_\mu + gA_\mu \right) dx^\mu = g\iint G_{\mu\nu} d\sigma^{\mu\nu}, \tag{49}$$

a concept which is missing entirely from Maxwell-Heaviside electrodynamics and which was introduced by Wu and Yang {31}. There is no reasonable doubt therefore, that electrodynamics is a non-Abelian gauge field theory as developed in this paper.

DISCUSSION

It has been argued in this paper that the Yang-Mills equations can be applied self-consistently to electrodynamics under all conditions, using an O(3) internal gauge field symmetry. Particular vacuum solutions of the field equations give rise to what are essentially novel angular momentum and energy relations between components of the complete field. The existence of the empirical observable $A^{(1)} \times A^{(2)}$ indicates the self-inconsistency of the procedure used to obtain the Maxwell-Heaviside equations from general gauge field theory, a self-inconsistency which is removed in Yang-Mills theory because the commutator of potentials no longer vanishes. The $B^{(3)}$ component of the Yang-Mills equations is a constant of motion and is essentially the fundamental spin of the electromagnetic field. The field equations in self-dual form have minimum action solutions which are instantons, and allow for the existence of low energy topological magnetic monopoles in electrodynamics. They also give a self-consistent description of the topological phase, the inverse Faraday effect and the third Stokes parameter on a classical level. The argument for an O(3) symmetry electromagnetic sector means that unified field theory has to be reconsidered. At present, it relies on the assumption that the electromagnetic sector has U(1) gauge field symmetry, and so becomes self-inconsistent because U(1) electrodynamics is self-inconsistent as argued in this paper and elsewhere {23-26}. Some work has been completed recently on an SU(2)×SU(2) symmetry electroweak theory {38} and on a non-Abelian quantum electrodynamics {39}, in which small corrections appear at fifth order in the fine structure constant to the electronic g factor and Lamb shift. These are in principle observable empirically, but are very small in magnitude. Therefore the extension from Abelian to non-Abelian in quantum electrodynamics has this effect theoretically.

The Yang-Mills equations developed in this paper were of course originally intended by these authors {17} as a generalization of electrodynamics, but this route seems to have been abandoned because of the acceptance of a U(1) sector symmetry, a linear and Abelian symmetry. This paper has set out to argue the inconsistencies in the U(1) sector symmetry and has attempted to develop the Yang-Mills equations anew by introducing and developing the $B^{(3)}$ concept in the vacuum. This development has shown that the $B^{(3)}$ field is responsible for the topological phases, the third Stokes parameter and magneto-optical effects. In the vacuum, the $B^{(3)}$ field introduces new angular momentum and energy relations. In showing that the $B^{(3)}$ field is a natural outcome of the Yang-Mills equations, the route is opened to concepts in high energy physics which can be adapted for electrodynamics.

An important philosophical consequence is that the potential becomes meaningful on a classical level as argued also by Barrett {30}. A clear example of this is that the $B^{(3)}$ field is defined in terms of a physically meaningful conjugate product $A^{(1)} \times A^{(2)}$ of potentials. The rules of gauge transformation in O(3) symmetry gauge field theory must be applied to the conjugate product regarded as physical construct, and to $B^{(3)}$, a physical magnetic field. The gauge transform then becomes a physical rotation which can be shown to be the cause of the Sagnac effect {35} when the platform is rotated. This procedure and understanding is quite different from the rules of gauge transformation applied to a potential in U(1) gauge field theory. In that case, an essentially arbitrary quantity is added to the original potential without affecting the magnetic field, which is regarded as the physical entity.

Therefore the Yang-Mills equations applied to electromagnetism, as originally intended, produce a subject which is philosophically quite different from the Maxwell-Heaviside view of electrodynamics. The Yang-Mills equations are gauge and Lorentz covariant and conserve the fundamental symmetries, such as C symmetry. These properties are well known because the Yang-Mills equations are foundational in contemporary particle physics. There seems no reason why they cannot be applied, as originally intended, to the electromagnetic sector, and systematic development in this direction is underway {26}.

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