THE MODERN WHITTAKER PAPERS

ON WHITTAKER'S REPRESENTATION OF THE ELECTROMAGNETIC FIELD IN TERMS OF TWO SCALAR POTENTIALS, PART I

ABSTRACT

It is argued that there exists in the classical vacuum longitudinal components of the electromagnetic field. In a U(1) symmetry gauge theory of electromagnetism, examples are Whittaker's standing waves, which emerge from an analysis of the electromagnetic field in terms of two scalar potentials; and Jackson's analysis of a modulated plane wave of finite lateral extent. In an O(3) symmetry gauge theory of electrodynamics, there exists a fundamental, phase free, field which is the fundamental spin, directed longitudinally in the axis of propagation. Therefore the view of electromagnetic waves in the radiation zone as being plane waves of infinite lateral extent is a mathematical ideal, a limited subset which excludes incorrectly the longitudinal components in vacuo. If the photon is assigned a finite mass, longitudinal waves in vacuo appear from the Proca equation or Higgs mechanism.

INTRODUCTION

The received view of classical electromagnetism {1, 2} in the radiation zone is given in terms of transverse plane waves of infinite lateral extent, with no longitudinal component. This concept is a mathematical solution of the Maxwell-Heaviside equations when the source terms are zero. Unfortunately, it has become habitual to refer to the concept of plane wave as physically meaningful in the radiation zone, and to refer to source free regions which in vacuo are a field (result) without source (cause). The received view also asserts that there can be no longitudinal components of electromagnetic radiation in the radiation zone. This widely accepted view is a consequence of the mathematical idealization of plane waves of infinite lateral extent. Jackson {3} has shown that if the electromagnetic field in the radiation zone is slowly modulated so that the transverse wave components are not infinite in lateral extent, longitudinal propagating waves appear in the vacuum in the radiation zone from the Maxwell-Heaviside equations. This conclusion has been reached repeatedly in this decade by several authors using very different methods of solution of the Maxwell-Heaviside equations and modifications thereof {4-12}.

The contemporary view that classical electromagnetism is a U(1) gauge theory {13} relies on the restricted received view of transverse plane waves, U(1) being isomorphic with O(2), the group of rotations in a plane. If there are longitudinal components available from the Maxwell-Heaviside equations themselves, then these cannot be represented by a U(1) gauge theory. It was shown nearly one hundred years ago by Whittaker {14, 15} that longitudinal standing waves exist in the vacuum from the Maxwell-Heaviside equations represented as the most general solutions possible of the d'Alembert wave equation. Jackson's well known demonstration of longitudinal vacuum waves also illustrates that the group O(2) in gauge theory must be replaced by the group O(3), that of rotations in three space dimensions, the covering group of SU(2). This realization leads in turn to the fact that classical electromagnetism is, according to gauge theory, a Yang-Mills theory, with an internal gauge space that is a vector space, rather than a scalar space as in the received Maxwell-Heaviside equations. Therefore, there is an iterative process at work here, and the Maxwell-Heaviside point of view (O(2) internal gauge symmetry) is the first step of the iteration. The Yang-Mills point of view (O(3) internal gauge symmetry), the second step of the iteration. There is plentiful empirical evidence at hand {16-20} to suggest that the first step in the iteration is not a complete description of classical electromagnetism; and that the second step is more accurate, as in any iterative process. The consequences of using a Yang-Mills theory for classical electromagnetism have been developed extensively by several authors {21-24}.

In section 2, the work of Whittaker {14, 15} is reviewed to show that there exists longitudinal standing wave solutions in general in the vacuum from the Maxwell-Heaviside equations. The work of Jackson {3} is reviewed to show that there also exist longitudinal propagating waves from the same equations. These examples show that the received O(2) gauge theory of classical electromagnetism must be replaced by an O(3) gauge theory, which gives a novel fundamental spin of the electromagnetic field discussed in section 3. The O(3) group is the covering group of SU(2), within which electromagnetism has been developed by Barrett {16, 17}. Symmetry breaking of SU(2) to O(3) with the Higgs field gives a view of electromagnetism similar to Whittaker's in terms of two scalar potentials, and this is developed in section 3 in work by Crowell. The discussion section develops the work by Whittaker in terms of novel ideas about the nature of the electromagnetic field.

LONGITUDINAL STANDING WAVES IN THE RADIATION ZONE

The work of Whittaker {14,15} is well summarized by Barrett {16}. Whittaker's development showed that the force potential can be defined in terms of both propagating and standing waves, the latter being longitudinal in the vacuum. He also showed that any electromagnetic field component can be expressed in terms of the derivatives of only two scalar potentials, and also be related to an inverse square law of attraction. He provided a general solution of the d'Alembert equation for a potential V in the vacuum of the form

$$V = \int_0^{2\pi} \int_0^{\pi} f(X \sin u \cos v + Y \sin u \sin v + Z \cos u + \frac{t}{k}, u, v)\, du\, dv \tag{1}$$

where f is an arbitrary function of the three arguments {14}

$$X \sin u \cos v + Y \sin u \sin v + Z \cos u + \frac{t}{k}, u, \text{ and } v,$$

where V is the potential also satisfying Laplace's equation at any point (X, Y, Z) of any distribution of matter of mass m, situated at the point (a, b, c). The solution (1) can be analyzed {16} into simple plane wave solutions and for any force varying as the inverse square of distance, the potential of such a force satisfies both the Laplace and d'Alembert equations in vacuo. The potential can be analyzed into simple plane waves propagating at a constant velocity, but the sum of these waves does not vary with time, they are standing waves in the vacuum which are longitudinally directed. Therefore V can be defined {14} in terms both of standing waves, which are global or non-local {16} and local propagating waves depending on time.

A simple example shows how Whittaker's general solution (1) can produce a longitudinally directed standing wave in the vacuum. This is a mathematical indication that the O(2) group or U(1) group cannot represent the electromagnetic field in the radiation zone.

If $u = 0$ and $\frac{1}{k} = c$, the vacuum speed of light, then

$$V = \int_0^{2\pi} \int_0^{\pi} f(Z + ct, 0, v)\, du\, dv \tag{2}$$

$$V = \pi \int_0^{2\pi} f(Z + ct, 0, v)\, dv \tag{3}$$

Now let $v = v_0$, a constant, then

$$V = 2\pi^2 f(Z + ct, 0, v_0). \tag{4}$$

Since f is an arbitrary function of its arguments, we may have:

$$V_1 = \sin(Z + ct). \tag{5}$$

Repeat with $\cos u = -1$ to obtain

$$V_2 = \sin(-Z + ct). \tag{6}$$

In the small angle limit:

$$V_1 \approx Z + ct \tag{7}$$

$$V_2 \approx -Z + ct. \tag{8}$$

This is a constant phaseless "standing wave" in the longitudinal (Z) direction, and so longitudinal solutions in the vacuum are possible of the d'Alembert wave equation. At a particular point (X, Y, Z),

$$V_1 - V_2 \approx 2Z \tag{9}$$

For proper units, we need:

$$\begin{aligned} V_1 &= \sin(\kappa Z + \omega t); \\ V_2 &= \sin(-\kappa Z + \omega t); \\ V_1 - V_2 &\approx 2\kappa Z; \end{aligned} \tag{10}$$

which is a standing wave in the Z direction with wavenumber κ. It does not depend on (or propagate with) time. Examples of propagating solutions from the general solution (1) may be given as follows:

$$\begin{aligned} V_3 &= e^{i(\omega t - \kappa Z)} \\ V_4 &= e^{-i(\omega t - \kappa Z)} \end{aligned} \tag{11}$$

A particular combination of these scalar potentials leads to the contemporary vector potentials:

$$\boldsymbol{A} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{12}$$

$$\boldsymbol{A}^* = \frac{A^{(0)}}{\sqrt{2}}(-i\boldsymbol{i} + \boldsymbol{j})e^{-i(\omega t - \kappa Z)} \tag{13}$$

In O(3) electrodynamics {21-24}, these potentials lead self-consistently to a longitudinal phaseless component of the field in vacuo:

$$\boldsymbol{B}^{(3)} \equiv -ig\boldsymbol{A} \times \boldsymbol{A}^* \equiv -ig\boldsymbol{A}^{(1)} \times \boldsymbol{A}^{(2)} \tag{14}$$

where g is a coefficient {21-24} that couples the field to its source, an electron.

The sine waves in eqn. (10) are generated inter alia by the reflection operation, so represent two waves moving in opposite directions, whose difference is a standing wave. In the small angle limit, i.e. $\sin(X) \approx X$, this standing wave is $2\kappa Z$, which is obviously directed in the propagation axis. This occurs in free space and is the result of the Maxwell-Heaviside equation, so the latter cannot be represented by a U(1) or O(2) gauge field symmetry in the radiation zone, as is the received view {13}. A gauge field symmetry is needed which is able to account for the existence in the vacuum of physical longitudinal waves, or the fundamental spin represented by eqn. (14), the result of a gauge field theory with an O(3) internal vector space, O(3) being the group of rotations in three dimensions. The $\boldsymbol{B}^{(3)}$ component {21-24} depends neither on time nor wavenumber, and so is a global quantity in Barrett's nomenclature {16}. It is the result of a theory of electrodynamics {16, 21-24} which is more self-consistent and comprehensive than the Maxwell-Heaviside theory, i.e. the result of the second stage in the iterative process toward an as yet unknown but yet more complete theory of electrodynamics. The U(1) theory is clearly untenable.

Another clear example of the fact that Maxwell-Heaviside electromagnetism is not a U(1) gauge theory is given by Jackson {3}. If, in the radiation zone, a circularly polarized wave moving in the Z direction has a finite extent in the X and Y directions, and is subjected to slowly varying amplitude modulation (the wave is many wavelengths broad), the electric and magnetic waves are given approximately by:

$$\boldsymbol{E}(X,Y,Z,t) \approx (E_0(X,Y)(\boldsymbol{i} \pm \boldsymbol{j}) + \frac{i}{\kappa}\left(\frac{\partial E_0}{\partial X} \pm i\frac{\partial E_0}{\partial Y}\right)\boldsymbol{k})e^{i\phi};$$
$$\phi \equiv \kappa Z - \omega t; \qquad \boldsymbol{B} \approx \mp\sqrt{\mu\varepsilon}\,\boldsymbol{E}; \tag{15}$$

which has a z component in the radiation zone in the vacuum. The group O(2), the covering group isomorphic with U(1), is on the other hand the group of rotations in two dimensions only (X and Y). The electromagnetic field in the radiation zone is therefore not represented by U(1) as in the received view {13}. The group space O(2) is a circle, whereas that of O(3), the covering group isomorphic with SU(2), is a sphere. The received view of the electromagnetic field in gauge theory {13} is that it is the gauge field that has to be introduced to guarantee invariance under local U(1) gauge transformation, isomorphic with O(2) transformations whose group space is a circle. However, if the group space is a circle, longitudinal solutions cannot exist in the radiation, and we have just shown the existence of such longitudinal solutions from the Maxwell-Heaviside equations. It is concluded that the Maxwell-Heaviside equations are not consistent with a U(1) gauge theory. The next gauge group, O(3), produces Yang-Mills equations which lead to the independently derived but isomorphic Barrett {16, 17, 25} and Evans {21-24} equations of the electromagnetic field, the former in SU(2), the latter in O(3). This is the result of the second step in the iterative process, new field equations with a much richer structure than the Maxwell-Heaviside equations {3, 13}. There are many phenomena now known {4-12, 16-24} that are not described by the Maxwell-Heaviside equations but are described by the isomorphic Barrett or Evans equations, or important equations due to Harmuth {26} and Lehnert {27} with a structure similar to them.

Whittaker at the end of his second paper {15} arrives at **general** equations defining the electromagnetic field in terms of purely longitudinal vector potentials $\boldsymbol{f}$ and $\boldsymbol{g}$ directed in the axis of propagation ($\boldsymbol{k}$) and defined by:

$$\boldsymbol{f} = F\boldsymbol{k}; \qquad \boldsymbol{g} = G\boldsymbol{k}; \tag{16}$$

and states that the electromagnetic field is defined by $\boldsymbol{f}$ and $\boldsymbol{g}$. The electric and magnetic components of the electromagnetic field under any circumstances are defined by Whittaker in S.I. units as follows:

$$\boldsymbol{E} = c\nabla \times (\nabla \times \boldsymbol{f}) + \nabla \times \dot{\boldsymbol{g}}; \qquad \boldsymbol{B} = \frac{1}{c}\nabla \times \dot{\boldsymbol{f}} - \nabla \times (\nabla \times \boldsymbol{g}); \tag{17}$$

and this includes source free regions.

If we use the contemporary Stratton four potential, defined by:

$$B^{\mu} \equiv (cP, \boldsymbol{S}) \tag{18}$$

where

$$\boldsymbol{E} = -\nabla \times \boldsymbol{S}; \qquad \boldsymbol{B} = -\frac{\partial \boldsymbol{S}}{\partial t} - \nabla P; \tag{19}$$

and the contemporary four potential, defined by:

$$A^{\mu} \equiv (\phi, c\boldsymbol{A}) \tag{20}$$

where

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}; \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi \tag{21}$$

It is straightforward to show that in S.I. units:

$$\boldsymbol{A} = -\nabla \times \boldsymbol{g} + \frac{1}{c}\dot{\boldsymbol{f}} \tag{22}$$

$$\boldsymbol{S} = -c\nabla \times \boldsymbol{f} - \dot{\boldsymbol{g}} \tag{23}$$

under all conditions, including the vacuum. So in general, the vector potential $\boldsymbol{A}$ and the Stratton potential $\boldsymbol{S}$ both have longitudinal components in the vacuum. These are physical. The transverse components of $\boldsymbol{A}$ and $\boldsymbol{S}$ are generated from the more fundamental $\boldsymbol{f}$ and $\boldsymbol{g}$ as shown in eqns. (22) and (23). The longitudinal components of $\boldsymbol{A}$ and $\boldsymbol{S}$ in the vacuum, usually overlooked in the contemporary literature are:

$$\begin{aligned} A_z &= \frac{1}{c}\dot{f}; & S_z &= -\dot{g}; \\ &= \frac{1}{c}\dot{F}; & &= -\dot{G}. \end{aligned} \tag{24}$$

The longitudinal magnetic and electric field components given by Whittaker under all conditions, including the vacuum, are:

$$B_z = \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2}; \qquad E_z = \frac{\partial^2 F}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}; \tag{25}$$

and these are again overlooked in the contemporary literature.

Whittaker's general theory is based on the d'Alembert equation and is still a U(1) symmetry theory in contemporary gauge theory classification. The U(1) gauge theory clearly produces longitudinal solutions under all conditions, including the vacuum, and the contemporary view that such vacuum solutions must be purely transverse is incorrect. Whittaker shows with precision that there are longitudinal solutions in general in the radiation zone. The concept of U(1) electrodynamics {13} cannot be interpreted to mean that all radiation is in the form of plane waves. The correct interpretation is that the Lagragian is invariant under a U(1) transformation. If so, there can be both transverse and longitudinal components as shown by Whittaker.

Whittaker regards F and G as physical properties of the electromagnetic field, each satisfying

$$\square F = 0; \qquad \square G = 0, \tag{26}$$

which are massless Klein-Gordon equations as well as being d'Alembert equations in the vacuum for F and G. These equations can be obtained from a Lagrangian of the form

$$\mathcal{L} \propto \partial_\mu F \partial^\mu G \tag{27}$$

and this Lagrangian is invariant under the U(1) gauge transformations:

$$F \rightarrow e^{-i\Lambda} F; \qquad G \rightarrow e^{i\Lambda} G \tag{28}$$

and therefore so are eqns. (26), whose general solution {14, 15} are

$$F = \int_0^\pi \int_0^{2\pi} f(X \sin u \cos v + Y \sin u \sin v + Z \cos u + ct, u, v)\, du\, dv \tag{29}$$

$$G = \int_0^\pi \int_0^{2\pi} g(X \sin u \cos v + Y \sin u \sin v + Z \cos u + ct, u, v)\, du\, dv \tag{30}$$

where f and g are **arbitrary** functions of their arguments. This arbitrariness accounts for the U(1) gauge transformation: both F and $e^{-i\Lambda}F$ have general solutions of the type (29).

Therefore we depart from the contemporary viewpoint once again, because changing F to $e^{-i\Lambda}F$ introduces a phase shift which is real and physical. This line of thought leads us directly to the concepts of contemporary non-Abelian gauge theory, where a reorientation in the internal gauge space induces a measurable and physical phase shift. In Whittaker's theory, we note that there are non-zero E_Z, B_Z, A_Z and S_Z in general in the radiation zone from a theory whose Lagrangian is invariant under a U(1) gauge transformation. The electromagnetic field is defined completely in terms of Whittaker's longitudinal vector potentials $\boldsymbol{f}$ and $\boldsymbol{g}$. The contemporary view that there cannot be longitudinal components in the radiation zone from a U(1) theory is therefore erroneous and confused as it is possible to be.

Whittaker anticipates therefore much of contemporary non-Abelian gauge theory applied to classical electrodynamics. The experimental effects of the longitudinal $\boldsymbol{f}$ and $\boldsymbol{g}$ are well known {28} but cannot be explained in the contemporary interpretation of the Maxwell-Heaviside equations. In the original equations of classical electrodynamics, due to J.C. Maxwell {29}, Faraday's electrotonic state is a physical vector potential, a term that was first introduced by Maxwell himself {30}. It is the later interpretation of J.C. Maxwell by Heaviside {31} that relegates the A^μ potential to a mathematical subsidiary without direct physical meaning. In O(3) electrodynamics (Section 3), Maxwell's original intent is restored.

Using plane waves for the transverse parts of A and S in the vacuum then in S.I. units:

$$\boldsymbol{S} = ic\boldsymbol{A}. \tag{31}$$

From eqns (22), (23), and (31), we obtain self-consistently:

$$\boldsymbol{f} = i\boldsymbol{g}; \qquad \dot{\boldsymbol{f}} = i\dot{\boldsymbol{g}}. \tag{32}$$

In order to obtain these transverse plane waves for $\boldsymbol{A}$ and $\boldsymbol{S}$, a possible solution for the function G is:

$$G = \frac{A^0}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)} \tag{33}$$

$$\boldsymbol{g} = G\boldsymbol{k} \tag{33a}$$

giving

$$\boldsymbol{A} = -\nabla \times \boldsymbol{g} = \frac{A^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{34}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{B^{(0)}}{\sqrt{2}}(i\boldsymbol{i} + \boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{34a}$$

Importantly, there also exists in the vacuum a **longitudinal** propagating part of the vector potential:

$$\boldsymbol{A}_L = \frac{i}{c}\dot{G}\boldsymbol{k} = -\kappa\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)}\boldsymbol{k} \tag{35}$$

which is overlooked in the contemporary texts {32}. For example, it is prohibited by the radiation and Coulomb gauges. The longitudinal vector potential $\boldsymbol{A}_L$ gives rise to the transverse magnetic plane wave:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}_L = -\frac{B^{(0)}}{\sqrt{2}}(\boldsymbol{i} + i\boldsymbol{j})e^{i(\omega t - \kappa Z)} \tag{36}$$

The electric field is defined by:

$$\boldsymbol{E}_L = \frac{\partial \boldsymbol{A}_L}{\partial t} - \nabla\phi = i\frac{\kappa^2}{c}\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)}\boldsymbol{k} - \nabla\phi \tag{37}$$

therefore in general, there is a longitudinal propagating component of the electric field in the vacuum, but using the relation:

$$\nabla\phi = \nabla \times \boldsymbol{S} - \frac{\partial \boldsymbol{A}}{\partial t} \tag{37a}$$

it is seen that the longitudinal part of $\nabla\phi$ is:

$$(\nabla\phi)_L = -\frac{\partial \boldsymbol{A}}{\partial t} \tag{37b}$$

so the net longitudinal propagating electric field is zero. Similarly, we have the magnetic field:

$$\boldsymbol{B}_L = -ic\frac{\partial \boldsymbol{A}_L}{\partial t} - \nabla P = \omega^2\frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)}\boldsymbol{k} - \nabla P \tag{38}$$

and using

$$\nabla P = \nabla \times \boldsymbol{A} + \frac{\partial \boldsymbol{S}}{\partial t} \tag{38b}$$

it is seen that the longitudinal part of ∇P is

$$(\nabla P)_L = \frac{\partial \boldsymbol{S}}{\partial t} \tag{39}$$

and the net longitudinal magnetic field in the vacuum is zero. These results are consistent with Whittaker's:

$$E_Z = \frac{\partial^2 F}{\partial Z^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = 0; \tag{40a}$$

$$B_Z = \frac{\partial^2 G}{\partial X^2} + \frac{\partial^2 G}{\partial Y^2} = 0 \tag{40b}$$

and the fact that Whittaker's theory is a U(1) gauge theory.

NON-ABELIAN ELECTRODYNAMICS

For several reasons, it is necessary to upgrade the Abelian U(1) structure used by Whittaker to a non-Abelian structure {4-12, 21-24} within the general framework of field theory {13}. One of the most compelling reasons is that the Maxwell-Heaviside equations, upon which the whole of Whittaker's analysis is based, fail to describe the Sagnac and Michelson effects {33}, while a theory of electrodynamics based on an O(3) internal structure succeeds in both case with precision. Barret {16, 17} has also independently developed equations of electrodynamics based on an SU(2) internal gauge symmetry which are isomorphic with the O(3) equations developed by Evans et al. {21-24}. Similar equations have been developed by Harmuth {26} and by Lehnert and Roy {27}. In O(3) electrodynamics {24} eqn. (14) defines the $\boldsymbol{B}^{(3)}$ field, which becomes:

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}(\nabla \times \boldsymbol{g}) \times (\nabla \times \boldsymbol{g}^*) \tag{41}$$

and can be expressed in terms of either G or F:

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\left(\frac{\partial G^*}{\partial Y}\frac{\partial G}{\partial X} - \frac{\partial G}{\partial Y}\frac{\partial G^*}{\partial X}\right)\boldsymbol{k} \tag{42}$$

$$\boldsymbol{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\left(\frac{\partial F^*}{\partial Y}\frac{\partial F}{\partial X} - \frac{\partial F}{\partial Y}\frac{\partial F^*}{\partial X}\right)\boldsymbol{k} \tag{42a}$$

The isomorphism between the Barrett field equations (16, 17) and those of Evans et al. (21-24) can be illustrated as follows (in work by L.W. Crowell) through the introduction of the Higgs field. The non-Abelian field tensor is in general {20-24}:

$$F_{ij}^a = \partial_j A_i^a - \partial_i A_j^a + \varepsilon^{abc}\left[A_i^b, A_j^c\right] \tag{43}$$

whose magnetic component is

$$B_i^a = \varepsilon_i^{\ jk} F_{jk}^a \tag{44}$$

where i, j, and k are restricted to spatial indices. We introduce the Higgs field:

$$H = (H^1, H^2, H^3) \tag{45}$$

with covariant derivative:

$$D_i H^a = \partial_i H^a + i\varepsilon^{abc} e\, A_i^b H^c \tag{46}$$

The Lagrangian for the gauge field is:

$$\mathcal{L}_G = -\frac{1}{4} F_{ij}^a F^{aij} \tag{47}$$

The action is the function dual to the 4-form $F \wedge F$ that determines the instanton number or topological charge k {33}:

$$8\pi^2 k = \int F^a \wedge F^a. \tag{48}$$

The Lagrangian for the Higgs field is:

$$\mathcal{L}_H = \frac{1}{2} D_i H^a D^i H^a + \frac{1}{2}\lambda\left(\left|H\right|^2 - m^2\right) \tag{49}$$

and the sum is:

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_H \tag{50}$$

We investigate the breaking of the SU(2) field (Barret equations) into the O(3) field (equations of Evans et alia). The classical action is the minimum of the Lagrangian, and fluctuations about this minimum represent quantum fluctuations. The minimum can occur at:

$$A^a = 0 \tag{51}$$

where the quartic potential term vanishes. If the Higgs potential vanishes:

$$H^3 = \sqrt{m}; \qquad H^1 = H^2 = 0 \tag{52}$$

This defines the ground state of the system where the massive H^3 restricts the vacuum from the symmetries of SU(2). The Higgs field is then a map from $\mathrm{SU}(2) \approx \mathrm{S}^3 \rightarrow \mathrm{S}^3$. Its action on the gauge field is to return functions of the gauge fields. We impose finite energy conditions on $|H|$ as $r \rightarrow \infty$. Further, the cross terms in the Lagrangian are bounded above by the minimization condition:

$$\left(B_i^a \pm D_i H^a\right)\left(B_i^a \pm D_i H^a\right) \geq 0. \tag{53}$$

The corresponding Hamiltonian to the Lagrangian then satisfies the condition:

$$H \geq \frac{1}{2}\lambda\left(|H|^2 - m^2\right) \pm B^{ai} D_i H^a. \tag{54}$$

Invoke the Taubes condition:

$$\frac{1}{2}\int B^{ai} D_i H^a = 4\pi\, n. \tag{55}$$

The total energy in the integral over the space-like manifold of the magnetic field is defined by:

$$E_n \geq \frac{1}{2}\lambda\left(|H|^2 - m^2\right) \pm 8\pi\, n \tag{56}$$

where equality occurs when:

$$B_i^a = \mp D_i H^a. \tag{57}$$

This is the Bogomolny condition. Now define the quantity:

$$Q = \int B_i^a H^a d^2 X^i \tag{58}$$

which is the Gauss Law in integral form for the calculation of charge within a region. Use the Divergence theorem:

$$\begin{aligned} Q &= \int B_i^a H^a d^2 X^i \\ &= \int \nabla_i (B_i^a H^a) d^3 X \\ &= \int B_i^a D^i H^a d^3 X = 8\pi\, n \end{aligned} \tag{59}$$

$$B_i^a = \varepsilon_i^{jk}\left(\partial_i A_k^a + i\varepsilon^{abc} A_i^b A_j^c\right). \tag{60}$$

For a = 1 and 2, the commutator implied by the Levi Civita symbol involves the A^3 field which is very massive {34, 35}, and vanishes. This leaves the standard curl terms in the integral, that sum to zero. For $a = 3$:

$$Q = -\int [A^1, A^2]^2 d^3 X \tag{61}$$

which implies that the commutator of the A^1 and A^2 fields are associated with a magnetic monopole field. At first order, a similar commutator defines the topological $\boldsymbol{B}$ field, (eqn. (21)). Further, the Bogomolny condition implies that the magnetic fields are the result of two scalar fields as inferred by Whittaker {14, 15}.

The derivation of eqn. (61) by Crowell is consistent with the definition {24} of the magnetic monopole as an area integral over $\boldsymbol{B}^3$:

$$Q = \frac{1}{V}\int B^{(3)} dAr. \tag{62}$$

To demonstrate the link between the two equations (61) and (62), we use the free space definition (from eqn. (21)):

$$B^{(0)} = \frac{e}{\hbar} A^{(0)2} \tag{63}$$

and the additional relations:

$$B^{(0)} = \frac{\omega}{c} A^{(0)}; \quad \hbar\omega = ecA^{(0)} = \frac{1}{\mu_0} B^{(0)2} V \tag{64}$$

where V is an integration volume {20-24}.

There are two surviving components, H^1 and H^2, of the Higgs field, giving the vector potentials:

$$\boldsymbol{A}^1 = \nabla H^1; \qquad \boldsymbol{A}^2 = \nabla H^2. \tag{65}$$

The $\boldsymbol{B}^{(3)}$ field is then

$$\boldsymbol{B}^{(3)} = -i\frac{e}{\hbar} \boldsymbol{A}^1 \times \boldsymbol{A}^2 = i\frac{e}{\hbar} \varepsilon_{ijk} \partial_j H^1 \partial_k H^2. \tag{66}$$

It is now assumed that

$$H^a = F_i^a e^i; \tag{67}$$

for $a = 1$ and 2. This gives:

$$\boldsymbol{B}^{(3)} = i\frac{e}{\hbar} \varepsilon_{ijk} \partial_j F_m^1 \partial_k F_n^2 e^m e^n. \tag{68}$$

Since e^m and e^n are orthogonal, their product can only be cyclic, so if $e^m e^n = \varepsilon^{mnr} e_r$:

$$\boldsymbol{B}^{(3)} = i\frac{e}{\hbar} e_{ijk} \varepsilon^{mnr} \partial_j F_m^1 \partial_k F_n^2 e_r \tag{69}$$

It is obvious that $r = i$, so the sum over k in the anti-symmetric symbols gives:

$$\boldsymbol{B}^{(3)} = i\frac{e}{\hbar} (\partial_{jm}\partial_{kn} - \partial_{jn}\partial_{km})(\partial_j F_m^1)(\partial_k F_n^2) \tag{70}$$

The first term is zero by symmetry and we obtain:

$$\boldsymbol{B}^{(3)} = i\frac{e}{\hbar} (\partial_j F_j^1 \partial_k F_k^2 - \partial_j F_k^1 \partial_k F_j^2) \tag{71}$$

which in this notation is eqn. (41).

Therefore a very important link has been established between the Higgs field and Whittaker's F in the context of non-Abelian electrodynamics.

DISCUSSION

On the U(1) level, there are longitudinal propagating solutions of the potential f and g, of the vector potential $\boldsymbol{A}$ and Stratton potential $\boldsymbol{S}$, but no longitudinal propagating components of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. So on the U(1) level, any physical effects of longitudinal origin in free space depend on whether or not f, g, $\boldsymbol{A}$ and $\boldsymbol{S}$ are regarded as physical or unphysical. Whittaker reduces the vacuum Maxwell-Heaviside equations to two d'Alembert equations in F and G, but it can always be claimed on the U(1) level that these are arbitrary {13}, and therefore not physically meaningful. On the O(3) level, the $\boldsymbol{B}^3$ field can be expressed directly and equivalently in terms of either F or G, which are therefore physical on this level. The longitudinal propagating potential $\boldsymbol{A}$ and Stratton potential $\boldsymbol{S}$ also becomes physical on the O(3) level.

There are claims in the literature that longitudinal propagating waves entities cause physical effects, such as the Priore effects {36}, which leads to beneficial medical consequences. The U(1) theory of electrodynamics does not give us much idea of where these effects are coming from, so they are usually dismissed as artifacts. However, there is an increasing amount of experimental evidence to suggest {36} that such effects do occur repeatedly and reproducibly in different laboratories. If they do occur, they are far easier to explain on the O(3) level, where potentials such as f, g, $\boldsymbol{A}$ and $\boldsymbol{S}$ are physical, and where there occur longitudinal vacuum fields such as $\boldsymbol{B}^3$. Such effects would tend to confirm that the non-Abelian electrodynamics is a more complete description of nature than its long accepted Abelian counterpart.

The main result of this paper is that all electromagnetic effects are derived from the time-like potential:

$$\phi = \dot{F} = i\dot{G} = -\omega \frac{A^{(0)}}{\sqrt{2}}(X - iY)e^{i(\omega t - \kappa Z)}. \tag{72}$$

The sum of ϕ and ϕ^* is real and physical:

$$\phi + \phi^* = -\frac{2\omega}{\sqrt{2}} A^{(0)}(X \cos\phi_0 + Y \sin\phi_0) \quad \text{where } \phi_0 \equiv \omega t - \kappa Z. \tag{73}$$

and the entities known as electric and magnetic fields are double differentials of ϕ. Canonical quantization of ϕ in the Lorenz gauge produces a physical time-like photon. The physical longitudinal photon is likewise obtained from ϕ/c. It is easily checked that the Lorenz condition used by Whittaker is obeyed:

$$\nabla \cdot \boldsymbol{A}_L + \frac{1}{c^2}\frac{\partial \phi_L}{\partial t} = 0. \tag{74}$$

So the conclusion reached here is the precise opposite of that reached in a contemporary gauge such as the radiation gauge, where ϕ is zero. It is also seen that if a U(1) gauge transformation is applied to ϕ:

$$\phi \rightarrow \phi + \frac{\partial X_L}{\partial t} \tag{75}$$

the magnetic and electric field components also change in general, so the function ϕ is not arbitrary as in the contemporary view.

In order to upgrade Whittaker to O(3), we can use the equations:

$$\square F = \square G = 0$$
$$\square F^* = \square G^* = 0 \quad (76)$$

to show that $\boldsymbol{B}^3$ is also defined by ϕ as in eqn (41).

There is definitive experimental evidence to show {36} that the longitudinal part of the electromagnetic entity reverses terminal tumours under strict protocol {36}. The electromagnetic entity causes de-differentiation in red blood cells, and tiny potentials are successful in healing otherwise intractable bone fractures. Cellular potentials, fields and waves are all represented by the time-like ϕ as a direct result of the work of Whittaker {14, 15}.

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