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On the nature of longitudinal solutions to Maxwell's equations in the Lorentz gauge

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Abstract

In the Lorentz gauge, Maxwell's equations provide four polarizations which can be interpreted as one time-like photon, two transverse space-like photons and one longitudinal space-like photon. It is shown that these solutions indicate the existence of novel longitudinal electric and magnetic fields, denoted $\mathbf{E}_{\text{II}}$ and $\mathbf{B}_{\text{II}}$ respectively, and which can be represented in terms of delta functions travelling at the speed of light. Thus $\mathbf{E}_{\text{II}}$ and $\mathbf{B}_{\text{II}}$ are novel, physically meaningful fields which are manifestly covariant and which are consistent, in the Lorentz gauge, with special relativity.

Introduction

It is well known that Maxwell's equations are Lorentz covariant and are consistent with the theory of special relativity. The electromagnetic field is an example of a gauge field (as opposed to a spinor field [1]) and its electric and magnetic components are customarily described as being invariant to the type of gauge used in its description, for example the Coulomb and Lorentz gauges. In the theory of special relativity, however, the scalar and vector potentials of the electromagnetic field form a four vector in pseudo-Euclidean space-time, and this imposes a general and fundamental restriction. The Coulomb gauge is not manifestly covariant [1] and is not rigorously consistent with special relativity. The Lorentz gauge, however, is manifestly covariant and may be consistent with special relativity. This leads to a little known but profound difficulty in the theory of electromagnetism which is described below, following Ryder [1].

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The electromagnetic field is usually assumed to have only two independent physically meaningful components, its right and left circular polarizations, which define the helicity of the photon, a "massless particle". These polarizations are transverse to the propagation axis of the plane wave. However, the electromagnetic field is described covariantly by a four-vector potential A_μ, with one time-like component and three space-like components, as for any four vector. It is therefore necessary to choose two of these as "physical" and discard the other two as "unphysical". This procedure loses manifest covariance [1]. If manifest covariance is retained rigorously as a fundamental physical law, all four components of A_μ must be physically meaningful, and A_μ is defined properly only in a gauge which retains all four components in this way. As shown by Ryder [1], this is the Lorentz gauge. In the Coulomb gauge only two components of A_μ are defined since the scalar potential (the time-like part of A_μ) is set to zero. The Coulomb gauge is therefore inconsistent with special relativity. This is easily seen from the fact that in special relativity, one must have (see Appendix):

$$\phi = |A| \tag{1}$$

where ϕ is the scalar potential and $|A|$ the vector potential's magnitude. In the Coulomb gauge, however, $\phi = 0$ and $|A| \neq 0$, so that Eq. (1) is immediately violated. In the Lorentz gauge:

$$\partial_\mu A^\mu = \frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot A = 0 \tag{2}$$

and condition (1) is not necessarily violated. Equations (1) and (2) must be satisfied simultaneously by ϕ and A in this gauge. Eq. (1) is a consequence of the fact that A_μ is a four vector in Minkowski space, and of the fact that the electromagnetic wave always travels at the speed of light c. Equation (1) is therefore a *fundamental* pseudo-Euclidean (i.e. geometrical) requirement of special relativity, but Eq. (2) is an assumption first made by Lorentz. This *assumption* defines the Lorentz gauge.

Any theory of electromagnetic radiation which is inconsistent with Eq. (1) must be inconsistent with the fundamentals of special relativity and should be discarded.

However, considerations of the Poincaré group [1] in the zero mass limit, clearly discussed in ref. 1, pp. 57–66, show that the state of a massless particle such as a photon is described by one number λ, the ratio of W^μ to P^μ; where W^μ is the Pauli–Lubanski pseudo-vector and where $P_\mu = i\partial/\partial\chi^\mu$ is the generator of translations. The number λ has the dimensions of *angular momentum* and is the helicity. With additional considerations of parity, the helicity takes on two values [1]; λ and $-\lambda$. We arrive at the well known result that the photon is a boson with helicity 1 and -1. Photons are then customarily described as being in right and left circularly polarized states with $\lambda = \pm 1$, but not $\lambda = 0$, since the latter is disallowed by the properties of the Poincaré group in special relativity.

In order to retain manifest covariance in the Lorentz gauge, however (or in any gauge that satisfies Eq. (1)), we must have *four* photon polarization states, and not two, i.e. we must consider all four components of A_μ in the description of the electromagnetic field, and regard all four as being physically meaningful. The only way out of this quandary is to realize that helicity can be related to photon polarization states *in more than one way*, i.e. there can be only two helicities for the photon, but these two can be related in more than one way to the four, manifestly covariant, states of polarization emerging from the d'Alembert equation

$$\Box A_\mu = 0 \tag{3}$$

in the Lorentz gauge [1]. Solutions of the d'Alembert equation are also solutions of the Maxwell equations in the Lorentz gauge, and are given by Ryder [1]. The first way of relating polarizations from the d'Alembert equation to the photon helicity is the well known one of accepting the transverse space-like photons as physically meaningful. There is, however, a second way which is introduced for the first time in this paper, and which shows that the Maxwell equations in the Lonentz gauge allow physically meaningful longitudinal solutions; a phase independent magnetic field, $\mathbf{B}_{\Pi}$, in the propagation axis, and a phase independent electric field $\mathbf{E}_{\Pi}$ in the same axis. These fields have been introduced elsewhere [2–5] and are shown in this paper to be rigorously consistent with special relativity and manifest covariance. They also provide the solution to the fundamental quandary that the massless gauge field can have four polarizations but only two helicities.

In this paper, the characteristics of the $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ fields are summarized in terms of creation and annihilation operators of the quantized field, and it is recalled [2–5] that the presence of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ does not affect the energy density of the electromagnetic wave. It is shown that this property is precisely that obtained from the Gupta–Bleuler condition in the quantization of the massless field in the Lorentz gauge. The Gupta–Bleuler condition also shows [1] that a combination of the time-like and longitudinal space-like photons of A_μ in the Lorentz gauge is a physically meaningful entity. The latter is identified precisely with the definitions of $\mathbf{E}_{\Pi}$ and $\mathbf{B}_{\Pi}$ in quantum field theory.

It is also recalled from previous work [2,3] that the magnetic field operator $\hat{B}_{\Pi}$ in the quantized field is directly proportional to the angular momentum operator $\hat{J}$ of the photon and thus to the two photon helicities. This completes the demonstration that the $\mathbf{B}_{\mathrm{II}}$ field can be identified with the time-like and longitudinal space-like polarizations of the electromagnetic field and that the $\mathbf{B}_{\mathrm{II}}$ field can be related to the two allowed photon helicities. It therefore becomes clear that the $\mathbf{B}_{\mathrm{II}}$ field links two of the allowed photon polarizations (time-like and longitudinal space-like) to the two allowed photon helicities. The transverse right and left circular polarizations link the other two photon polarizations to the photon helicities. This conclusion is sketched as follows:

| Time-like polarization (0)
| Longitudinal space-like polarization (3) | ↘

Photon helicities
$\lambda = \pm 1$

| Left transverse space-like polarization (1) | ↗
| Right transverse space-like polarization (2) |

We have therefore solved the quandary of the massless gauge field's four polarizations and two helicities through the existence of a hitherto unidentified, but physically meaningful, $\mathbf{B}_{\mathrm{II}}$ field. The $\mathbf{E}_{\mathrm{II}}$ field completes the picture in that its existence is implied by that of the $\mathbf{B}_{\mathrm{II}}$ through our previously derived relation [5]:

$$\boldsymbol{B}_{\mathrm{II}} \times \boldsymbol{E} = \boldsymbol{E}_{\mathrm{II}} \times \boldsymbol{B} \tag{4}$$

where $\boldsymbol{B}(\boldsymbol{r}, t)$ and $\boldsymbol{E}(\boldsymbol{r}, t)$ are the usual oscillating fields of the electromagnetic wave.

The paper closes with a brief discussion of the physical phenomena expected from the existence of $\mathbf{B}_{\mathrm{II}}$ and $\mathbf{E}_{\mathrm{II}}$.

Longitudinal and time-like photons

It is necessary to recall, following ref. 1, pp. 148–153, that the quantization of the electromagnetic field in the Lorentz gauge proceeds through the Gupta–Bleuler condition:

$$\partial_\mu \hat{A}^{(+)\mu}|\psi\rangle = 0 \tag{5}$$

where $\hat{A}^{(+)\mu}$ is now an operator and where $|\psi\rangle$ is an eigenstate of the photon field. The condition (5) replaces the d'Alembertian operator condition

$$\Box \hat{A}_\mu = 0 \tag{6}$$

for the reasons described by Ryder [1]. The Gupta–Bleuler condition leads to

$$[\hat{a}^{(0)}(k) - \hat{a}^{(3)}(k)]|\psi\rangle = 0 \tag{7}$$

where $\hat{a}^{(0)}$ and $\hat{a}^{(3)}$ are annihilation operators corresponding to the time-like polarization (superscript (0)) and the longitudinal space-like polarization (superscript (3)). In turn, condition (7) leads to the expectation value equality

$$\langle\psi|\hat{a}^{(0)+}(k)\hat{a}^{(0)}(k)|\psi\rangle = \langle\psi|\hat{a}^{(3)+}(k)\hat{a}^{(3)}(k)|\psi\rangle \tag{8}$$

for any physical state $|\psi\rangle$ of the quantized electromagnetic field in the Lorentz gauge. The equality (8) contains products of annihilation and creation operators. Two points are important in this context: (i) there are four photon polarization states, labeled $\epsilon^{(0)}$ to $\epsilon^{(3)}$; (ii) the time-like and longitudinal space-like photons do not contribute to the electromagnetic energy density, but a combination, or *admixture*, of time-like and longitudinal space-like photons is a physically meaningful state [1]. The contributions of the state (0) and (3) photons to the electomagnetic energy density cancel, but, nevertheless, admixtures of these photon states are physically meaningful. This is the standard interpretation [1] of the Gupta–Bleuler condition of the quantized field, a condition which is derived, in turn, from the quantized d'Alembertian.

We have shown elsewhere [5] that there exist magnetic and electric field operators of the quantized electromagnetic field defined by

$$\begin{aligned} \hat{\boldsymbol{B}}_{\mathrm{II}} &= \tfrac{1}{2} B_0(\hat{a}_1^{(1)}\hat{a}_2^{(2)+} - \hat{a}_2^{(2)}\hat{a}_1^{(1)+})\boldsymbol{k} \\ \hat{\boldsymbol{E}}_{\mathrm{II}} &= -\tfrac{1}{2} E_0(\hat{a}_1^{(1)}\hat{a}_2^{(2)+} - \hat{a}_2^{(2)}\hat{a}_1^{(1)+})\boldsymbol{k} \end{aligned} \tag{9}$$

where $\boldsymbol{k}$ (not to be confused with the wave vector) is a unit axial vector for the magnetic field and a unit polar vector for the electric field, directed in both cases in the axis of propagation, Z, of the laboratory frame (X, Y, Z). Here $\mathbf{B}_{\mathrm{II}}$ and $\mathbf{E}_{\mathrm{II}}$ respectively are magnetic and electric field strength scalar amplitudes, and $\hat{a}$ and $\hat{a}^+$ are annihilation and creation operators. Both fields [5] are longitudinal solutions in free space of Maxwell's equations and both are defined in terms of the operator $(\hat{a}_1^{(1)}\hat{a}_2^{(2)+} - \hat{a}_2^{(2)}\hat{a}_1^{(1)+})$ whose expectation value between photon states is 2:

$$\langle\psi|\hat{a}_1^{(1)}\hat{a}_2^{(2)+} - \hat{a}_2^{(2)}\hat{a}_1^{(1)+}|\psi\rangle = 2 \qquad (10)$$

Furthermore, we have shown that $\mathbf{B}_{\mathrm{II}}$ and $\mathbf{E}_{\mathrm{II}}$ do not contribute to the electromagnetic energy density [5]. It is therefore reasonable to write the equation

$$\begin{aligned}\langle\psi|\hat{a}^{(0)+}(k)\hat{a}^{(0)}(k)|\psi\rangle &= \langle\psi|\hat{a}^{(3)+}(k)\hat{a}^{(3)}(k)|\psi\rangle \\ &= \langle\psi|\hat{a}_1^{(1)}\hat{a}_2^{(2)+} - \hat{a}_2^{(2)}\hat{a}_1^{(1)+}|\psi\rangle \\ &= -\langle\psi|\hat{a}_2^{(2)}\hat{a}_1^{(1)+} - \hat{a}_1^{(1)}\hat{a}_2^{(2)+}|\psi\rangle\end{aligned} \qquad (11)$$

which defines the novel fields $\mathbf{B}_{\mathrm{II}}$ and $\mathbf{E}_{\mathrm{II}}$ in terms of the expectation values between photon eigenstates of the operator products corresponding to time-like and longitudinal space-like annihilation and creation operators of the quantized field in the Lorentz gauge.

It is also possible to show [5] that the Stokes operator $\hat{S}_3$ [6] is defined as

$$\hat{S}_3 = \frac{-E_0^2}{2}(\hat{a}_1^{(1)}\hat{a}_2^{(2)+} - \hat{a}_2^{(2)}\hat{a}_1^{(1)+}) \qquad (12)$$

and therefore the expectation values (11) are identified as defining this well known operator, whose expectation value is the third Stokes parameter of the circularly polarized electromagnetic field.

The present authors have also shown [5] that the classical fields $\mathbf{E}_{\mathrm{II}}$ and $\mathbf{B}_{\mathrm{II}}$ (expectation values of $\hat{\boldsymbol{E}}_{\mathrm{II}}$ and $\hat{\boldsymbol{B}}_{\mathrm{II}}$ in photon states) can be described in terms of delta functions as

$$\begin{aligned}\boldsymbol{B}_{\mathrm{II}} &= \frac{B_0}{2\pi}\boldsymbol{k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{i\kappa(Z-ct)}\,\mathrm{d}\kappa\,\mathrm{d}Z \\ &\equiv B_0\boldsymbol{k}\int_{-\infty}^{\infty}\delta(Z-ct)\,\mathrm{d}Z\end{aligned} \qquad (13)$$

It is possible to show that this is precisely what is expected from the solution given by Ryder [1] for the quantized d'Alembertian operator condition (6), a solution which is given [1] as

$$\begin{aligned}\hat{A}_\mu(Z) &= \int\frac{\mathrm{d}^3k}{2(2\pi)^3k_0}\sum_{\lambda=0}^{3}\epsilon_\mu^{(\lambda)} \\ &\times[\hat{a}^{(\lambda)}(k)\mathrm{e}^{-ikZ} + \hat{a}^{(\lambda)+}(k)\mathrm{e}^{ikZ}]\end{aligned} \qquad (14)$$

This is clearly a sum over the four possible polarizations in Minkowski four space, a solution which is manifestly covariant. It is clear from Eq. (14) that if it were possible to write the annihilation and creation operators as

$$\hat{a}^{(\lambda)}(k) = \hat{a}^{(\lambda)}(0)\exp(ikZ_0) \qquad (15a)$$

$$\hat{a}^{(\lambda)+}(k) = \hat{a}^{(\lambda)+}(0)\exp(-ikZ_0) \qquad (15b)$$

the solution becomes a simple sum over delta functions, because, by definition

$$\delta(Z-Z_0) = \frac{1}{(2\pi)}\int \mathrm{e}^{ik(Z-Z_0)}\,\mathrm{d}k \qquad (16)$$

is the Dirac delta function. However, it is always possible to write Eq. (15) because by definition [4,7]:

$$\hat{a}^{(\lambda)}(t) = \hat{a}^{(\lambda)}(0)\exp(-i\omega t) \qquad (17a)$$

$$\hat{a}^{(\lambda)+}(t) = \hat{a}^{(\lambda)+}(0)\exp(+i\omega t) \qquad (17b)$$

where ω is the angular frequency at an instant t. As discussed, for example, by Kielich et al. [8], it is always possible to make the replacement

$$t = \frac{-Z_0}{c} \qquad \text{i.e. } Z_0 = -ct \qquad (18)$$

so that, using the definition $kc = \omega$, Eqs. (15a and b) become Eqs. (17a and b). The solution (14) therefore becomes a sum over delta functions:

$$\begin{aligned}\hat{A}_\mu(Z) &= \frac{1}{2k_0}\sum_{\lambda=0}^{3}[\epsilon_\mu^\lambda\hat{a}^{(\lambda)}(0)\delta(Z_0-Z) \\ &+ \epsilon_\mu^\lambda\hat{a}^{(\lambda)+}(0)\delta(Z-Z_0)]\end{aligned} \qquad (19)$$

This is the most general solution in special relativity of the quantized d'Alembertian; in other words, the most general solution for the potential four vector in relativistic quantum field theory [1]. The delta function solutions for the scalar potential corresponding to $\mathbf{E}_{II}$ and $\mathbf{B}_{II}$ have been shown by the present authors to be delta functions

$$\phi_\pi = \phi_0 \int_{-\infty}^{\infty} \delta(Z - ct)\,dZ \qquad (20)$$

and the vector potential corresponding to $\mathbf{E}_{II}$ and $\mathbf{B}_{II}$ to be

$$A_{II} = A_0 \int_{-\infty}^{\infty} \delta(Z - ct)\,dZ \qquad (21)$$

and clearly, these are special cases of the general solution (19).

It is concluded that $\mathbf{E}_{II}$ and $\mathbf{B}_{II}$ are rigorously consistent with relativistic quantum field theory in the Lorentz gauge.

The link between polarization and helicity

In this section we address the difficulty in massless gauge field theory that there are four polarizations and two helicities, and show that the difficulty disappears through the use of the magnetic field operator $\hat{\boldsymbol{B}}_{II}$.

Using Eqs. (9) and (12) it is clear that there is a relation between $\hat{\boldsymbol{B}}_{II}$ and the third Stokes operator $\hat{S}_3$:

$$\hat{\boldsymbol{B}}_{II} = -\frac{\hat{S}_3}{E_0 c}\boldsymbol{k} \qquad (22)$$

Furthermore, it is well known that the Stokes operators $\hat{S}_1$, $\hat{S}_2$, $\hat{S}_3$ obey the commutator relations of angular momentum in quantum field theory [9], showing that $\hat{\boldsymbol{B}}_{II}$ has the properties of quantized angular momentum of the electromagnetic wave. Standard theory [9] shows that

$$\langle\psi|\hat{S}_3|\psi\rangle = \langle\hat{\sigma}_z\rangle = \frac{\langle\hat{J}_z\rangle}{\hbar} \qquad (23)$$

where $\hat{\sigma}_z$ is the Pauli matrix operator

$$\hat{\sigma}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad (24)$$

The angular momentum operator $\hat{J}_z$ is defined by

$$\langle\psi|\hat{J}_z|\psi\rangle = \hbar S_3 \qquad (25)$$

and using Eq. (24) we obtain

$$\hat{\boldsymbol{B}}_{II} = B_0^{(1)}\frac{\hat{J}_z}{\hbar}\boldsymbol{k} = B_0^{(1)}\frac{\hat{\boldsymbol{J}}}{\hbar} \qquad (26)$$

This equation shows that the expectation value of the *longitudinal* magnetic field $\hat{\boldsymbol{B}}_{II}$ is the same as that of $\hat{\boldsymbol{J}}_z$, the angular momentum operator, between photon eigenstates $|\psi\rangle$, which in turn is the classical third Stokes parameter multiplied by $\hbar$. The expectation value of $\hat{J}_z$ is S_3 units of quantized angular momentum for *any* eigenstate $|\psi\rangle$ of the quantized field. If the beam were to consist of one photon of energy $\hbar\omega$, then $|\psi\rangle = |1\rangle$ and $\langle 1|\hat{J}|1\rangle$ would be the expectation value of $\hat{J}$ for one photon. In this case $\hat{B}_{II}$ would be the magnetic flux density operator of one photon, whose scalar magnitude we denote $B_0^{(1)}$.

Equation (26) therefore relates $\hat{\boldsymbol{B}}_{II}$ to the helicity of the photon, because the eigenvalues of the angular momentum operator $\hat{J}$ of one photon are $M_J\hbar$, where M_J takes the values of +1 and −1, but not zero, in precise analogy with the helicity λ.

It is clear therefore, as described in the introduction, that $\hat{\boldsymbol{B}}_{II}$ relates the time-like and longitudinal space-like photon polarizations to the helicity, and explains the fact that the massless electromagnetic gauge field has four polarizations but only two helicities.

Lastly, we recall [5] that the existence of $\mathbf{B}_{II}$ implies the existence of $\mathbf{E}_{II}$ through the equation:

$$\boldsymbol{B}_{II} \times \boldsymbol{E} = \boldsymbol{E}_{II} \times \boldsymbol{B} \qquad (27)$$

which was derived from considerations [5] of the continuity equation [7] linking the electromagnetic energy density and the Poynting vector in the presence of $\mathbf{B}_{II}$ and $\mathbf{E}_{II}$. These considerations show [5] that $\mathbf{B}_{II}$ and $\mathbf{E}_{II}$ do not contribute to the electromagnetic energy density. This is the precise counterpart of the finding that the hamiltonian operator in the quantized relativistic field [1] is proportional to an integral over the sum

$$\sum_{\lambda=1}^{3}(\hat{a}^{(\lambda)+}\hat{a}^{(\lambda)} - \hat{a}^{(0)+}\hat{a}^{(0)}) \qquad (28)$$

so that the contributions of the longitudinal and time-like photons cancel, leaving only the contributions from the transverse space-like photons, superscripted (1) and (2) in the sum (28). These contributions correspond to the right and left states of the classical oscillating and transverse fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$.

Discussion

If $\mathbf{B}_{II}$ is a physically meaningful field, as argued in this paper, then it must produce measurable effects. For example, circularly polarized electromagnetic radiation is able to magnetize material through the intermediacy of $\mathbf{B}_{II}$ and terms to higher order in $\mathbf{B}_{II}$. This is supported by experimental evidence such as the well known inverse Faraday effect [10] and the recently predicted [11] and measured [12] phenomenon of optical NMR spectroscopy, in which a circularly polarized visible frequency laser shifts NMR frequencies through $\mathbf{B}_{II}$ and higher order terms in $\mathbf{B}_{II}$. However the evidence to date is based on a small number of experimental investigations and is incomplete and rather fragmentary. Evans and co-workers [3,13–15] have suggested a number of tests for $\mathbf{B}_{II}$ based on the classical and semi-classical descriptions, in which $\mathbf{B}_{II}$ is proportional simply to the square root of the laser's intensity. (This simple first theory takes no account of statistical effects and effects due to inhomogeneous beam profiles and scattering and simply treats $\mathbf{B}_{II}$ as being "equivalent" to a regular magnetostatic field.) Among the effects due to $\mathbf{B}_{II}$ are: (i) the optical Faraday effect; (ii) the optical Zeeman effect; (iii) optically induced shifts in ESR spectra; (iv) the optical Cotton–Mouton effect: (v) optically induced forward–backward birefringence. Among the effects predicted for the vector and scalar potentials $\mathbf{A}_{II}$ and ϕ_{II} is an optical Aharonov–Bohm effect. The semi-classical predictions of most of these and details thereof are already available in the literature [3,13–15]. There are probably several other effects due to $\mathbf{B}_{II}$ which have not yet been predicted.

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Appendix: Eq. (1) of the text

In this Appendix we discuss the origin of Eq. (1) of the text, which shows that in a theory which retains manifest covariance, the magnitude of the scalar potential is equal to that of the vector potential.

It is well known that in special relativity, the divergence of a four vector is invariant under Lorentz transformation. In the summation convention:

$$\frac{\partial A'_\mu}{\partial \chi'_\mu} = \frac{\partial A_\mu}{\partial \chi_\mu} \tag{A1}$$

This condition is equivalent in three-dimensional notation to a continuity equation. Thus, if the four potential is defined by

$$A_\mu \equiv (A, i\phi) \tag{A2}$$

Eq. (A1) implies the continuity equation

$$\nabla \cdot A + \frac{1}{c}\frac{\partial \phi}{\partial t} = \text{constant} \tag{A3}$$

(constant = 0 for the Lorentz gauge)

which is the Lorentz condition, Eq. (2) of the text, defining the Lorentz gauge. The latter is evidently consistent with special relativity. If, in the Lorentz gauge, we assume that the scalar potential, ϕ, is zero, then the continuity Eq. (A3) gives the result:

$$\nabla \cdot A = 0 \tag{A4}$$

which is usually taken to define the Coulomb gauge. This appears, at this stage of the argument, still to be consistent with special relativity and the principle of covariance of physical laws. However, the four potential is a four vector in Minkowski space-time, and therefore its four components are subject to the restrictions of pseudo-Euclidean geometry.

To illustrate these, we provide several examples in this Appendix.

The interval in space-time of special relativity is well known to be defined by

$$ds^2 \equiv c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{A5}$$

At the speed of light, c, the universal constant, the interval ds^2 vanishes. This means that

$$dr^2 \equiv dx^2 + dy^2 + dz^2 = c^2 dt^2 \tag{A6}$$

and if a four vector is defined as

$$x_\mu \equiv (\boldsymbol{r}, ict) \tag{A7}$$

then, at the speed of light, we have

$$dr^2 = c^2 dt^2$$

$$|r| = ct \tag{A8}$$

i.e. *the magnitude of the scalar component of the four vector becomes equal, at the speed of light, to the magnitude of its vector component.*

Note that the continuity equation is obeyed for all ds^2, i.e.

$$\nabla \cdot r + \frac{1}{c}\frac{\partial (ct)}{\partial t} = \text{constant} \tag{A9}$$

This is our first example of the reasoning upon which we have based Eq. (1) of the text.

Example 2 is the momentum/energy four vector:

$$p_\mu \equiv \left(\boldsymbol{p}, i\frac{En}{c}\right) \tag{A10}$$

whose continuity equation is

$$\nabla \cdot \boldsymbol{p} + \frac{1}{c^2}\frac{\partial (En)}{\partial t} = 0 \tag{A11}$$

At the speed of light, special relativity gives the well known result

$$|\boldsymbol{p}| = \frac{En}{c} \tag{A12}$$

showing immediately that the magnitude of the vector part of the four vector becomes equal, as in our first example, to that of the scalar part.

Example 3 is the current density four vector

$$\boldsymbol{J}_\mu \equiv (\boldsymbol{J}, ic\rho) = (\rho \boldsymbol{v}, i\rho c) \tag{A13}$$

whose continuity equation is

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \tag{A14}$$

and, at the speed of light, it is clear that

$$|\boldsymbol{v}| = c \qquad |\rho \boldsymbol{v}| = |\boldsymbol{J}| = \rho c \tag{A15}$$

Example 4 is the electromagnetic energy density four vector

$$U_\mu \equiv \left(\boldsymbol{S}, i\frac{U}{c}\right) \tag{A16}$$

in which $\boldsymbol{S}$ is the Poynting vector and U the electromagnetic energy density. Its continuity equation is the well known

$$\nabla \cdot \boldsymbol{S} + \frac{1}{c^2}\frac{\partial (U)}{\partial t} = 0 \tag{A17}$$

and at the speed of light, it is clear that

$$|\boldsymbol{S}| = \frac{U}{c} \tag{A18}$$

Since electromagnetic radiation always travels at the speed of light, Eq. (A18) is always operative. This can be confirmed independently by noting that the Poynting vector in SI units is defined [9] as

$$S \equiv \frac{1}{\mu_0} E \times B^* \tag{A19}$$

and that the electromagnetic energy density is defined as

$$U = \tfrac{1}{2}\left(\epsilon_0 E \cdot E^* + \frac{1}{\mu_0} B \cdot B^*\right) \tag{A20}$$

Example 5 is the electromagnetic wavevector, angular frequency four vector:

$$\kappa_\mu \equiv \left(\kappa, i\frac{\omega}{c}\right) \tag{A21}$$

whose continuity equation is

$$\nabla \cdot \kappa + \frac{1}{c^2}\frac{\partial \omega}{\partial t} = 0 \tag{A22}$$

and, at the speed of light

$$|\kappa| = \frac{\omega}{c} \tag{A23}$$

which is the usual definition of κ in free space. Again, the scalar part of the four vector is equal to the magnitude of the vector part. This conclusion is also consistent with the well known (Eq. (A12)):

$$En = \hbar\omega \qquad p = \hbar\kappa \tag{A24}$$

Therefore, in each case, there exists a relation at the speed of light which equates the magnitude of the space-like part of the four vector to the magnitude of the time-like part.

Since the four potential is a four vector, it must also obey this relation at the speed of light. Since the electromagnetic wave always travels in free space at the speed of light, it follows that in free space

$$\phi = |A| \tag{A25}$$

which is Eq. (1) of the text. Clearly, if $\phi = 0$, as in the Coulomb or radiation gauge, then this means that A must be zero in the Coulomb gauge. If not, the theory is inconsistent with special relativity.

It has been shown elsewhere [3,15] that the longitudinal field $\mathbf{B}_{\text{II}}$ of the text obeys the continuity equation:

$$\nabla \cdot B_{\text{II}} + \frac{\partial U_{\text{II}}}{\partial t} = 0 \tag{A26}$$

where $U_{\text{II}} = -B_0/c$ (in SI units) is the magnetic density of the electromagnetic wave. Here

$$B_{\text{II}} \equiv \frac{E \times E^*}{(2E_0 ci)} \tag{A27}$$

and

$$U_{\text{II}} = \frac{1}{2E_0 ci}\left(\int E^* \cdot \frac{\partial B}{\mathrm{d}t}\mathrm{d}t - \int E \cdot \frac{\partial B^*}{\partial t}\mathrm{d}t\right) \tag{A28}$$

Therefore, there exists the four vector

$$B_{\text{II}\mu} \equiv (B_{\text{II}}, iU_{\text{II}}c) \tag{A29}$$

and at the speed of light

$$U_{\text{II}}c = -B_0 = |B_{\text{II}}| \tag{A30}$$

since $|\mathbf{B}_{\text{II}}|$ can be $\mp B_0$

Importantly, this shows that the concept of $\mathbf{B}_{II}$ is consistent with special relativity, as described also in the text of this paper. This supports the fact that $\mathbf{B}_{\text{II}}$ is physically meaningful, and that $\mathbf{E}_{\text{II}}$ is also physically meaningful.

$\mathbf{B}_{\text{II}}$ and $\mathbf{E}_{\text{II}}$ form the four vector

$$B_{\text{II}\mu} \equiv \left(B_{\text{II}}, -i\frac{E_0}{c}\right) \qquad E_{\text{II}\mu} \equiv (E_{\text{II}}, -icB_0) \tag{A31}$$

in SI units, and at the speed of light is seen that the scalar and vector components become equal in magnitude. The continuity equation from Eq. (A31) is the Maxwell equation

$$\nabla \cdot B_{\text{II}} = \frac{1}{c^2}\frac{\partial E_0}{\partial t} = 0 = \nabla \cdot E_{\text{II}} = \frac{\partial B_0}{\partial t} \tag{A32}$$

showing that $\mathbf{B}_{\text{II}}$ and $\mathbf{E}_{\text{II}}$ obey a Maxwell equation and are therefore magnetic and electric fields.

Finally, the four-dimensional Laplace equation is

$$\Box^2 A_\mu \equiv \frac{\partial^2 A_\lambda}{\partial x_\lambda^2} = 0 \tag{A33}$$

which in three dimensions is

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)A_\mu = 0 \tag{A34}$$

If we have $\phi = |\mathbf{A}|$ at the speed of light, Eq. (A34) implies that

$$\nabla^2 \phi = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} \tag{A35}$$

$$\nabla^2 A = \frac{1}{c^2} \frac{\partial^2 A}{\partial t^2} \tag{A36}$$

These equations are the well known equations whose solutions are the Liénard–Wiechert potentials. Therefore the condition $\phi = |A|$ is consistent with these potentials.