APPENDIX D: THE CONSTANT OF INTEGRATION IN EQUATION (35)

Most generally, from Eq. (35),

$$\mathbf{E}_\Pi \times \mathbf{B} = \mathbf{B}_\Pi \times \mathbf{E} + \text{constant} \tag{D.1}$$

The dual transformation of special relativity means that $\mathbf{E}_\Pi$ and $-(c/\mathrm{i})\mathbf{B}_\Pi$, for example, are indistinguishable solutions of Maxwell's equations; i.e., it is possible to replace $\mathbf{E}_\Pi$ everywhere by $-(c/\mathrm{i})\mathbf{B}_\Pi$ without changing the Maxwell equations, and therefore without changing the solutions to the equations. The dual transformation, however, does not affect the constant in Eq. (D.1), which is independent of $\mathbf{E}$, $\mathbf{B}$, $\mathbf{E}_\Pi$, and $\mathbf{B}_\Pi$. Thus, applying the dual transform,

$$\mathbf{B}_\Pi \times \mathbf{E} = \mathbf{E}_\Pi \times \mathbf{B} + \text{constant} \tag{D.2}$$

Adding Eqs. (D.1) and (D.2) yields

$$\text{Constant} = 0$$

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ON LONGITUDINAL FREE SPACETIME ELECTRIC AND MAGNETIC FIELDS IN THE EINSTEIN-DE BROGLIE THEORY OF LIGHT

I. INTRODUCTION

It is usually concluded in electrodynamical literature[1–16] that the photon is massless and that the range of the electromagnetic field is infinite. This conclusion is not, however, supported by experimental data. To the contrary, Vigier[17] has recently reviewed a substantial amount of evidence that leads to the conclusion of finite photon rest mass. These data include, to take two of many examples, the direction-dependent anisotropy of the frequency of light in cosmology and frequently observed anomalous red shifts.

In papers and correspondence circa 1916 to 1919,[17] Einstein[18] proposed a photon rest mass[19] that can be estimated from the Hubble constant to be about 10^{-68} kg. An immediate consequence is that the d'Alembert equation is replaced by the Einstein-de Broglie-Proca (EBP) equation, which can be expressed[20, 21] in the form

$$\Box A_\mu = -\xi^2 A_\mu \tag{1}$$

where

$$\xi = \frac{m_0 c}{\hbar}$$

Here m_0 is the photon rest mass, c the speed of light, the universal constant of special relativity, and $\hbar$ the reduced Planck constant. The potential four vector A_μ of the de Broglie-Proca field is manifestly covariant, and has four, physically meaningful, components, one timelike ((0)) and three spacelike, of which two are transverse ((1) and (2)) and one is longitudinal ((3)). From Eq. (2), the range ξ^{-1} of the field becomes 10^{26} m, cosmic in dimensions, but finite. Equation (1) is an expression of the Einstein-de Broglie theory of light[17] and implies that gauge transformations of the first and second kind[20, 21] can no longer be interpreted as implying zero photon rest mass. It is well known that the EBP equation implies mathematically[20, 21] the Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = 0 \tag{2}$$

for the massive boson. If the photon has rest mass, it is always described by the Lorentz condition. Experimental evidence[17] for finite photon rest mass implies that gauge invariance must be reinterpreted fundamentally, and this is part of the purpose of this paper, in which it is shown that finite m_0 is consistent with gauge invariance of the first and second kind if and only if

$$A_\mu A_\mu = 0, \quad m_0 \neq 0 \tag{3a}$$

a condition that implies

$$\phi = c|\mathbf{A}| \tag{3b}$$

where

$$A_\mu = \left(\mathbf{A}, \frac{i}{c}\phi\right) \tag{3c}$$

and ϕ is the scalar potential and $\mathbf{A}$ the vector potential of the de Broglie-Proca field. Condition (3a) is consistent with the Lorentz condition (2), but is inconsistent with a massless gauge such as the traditional Coulomb gauge.[1–16]

Furthermore, the notion of zero photon rest mass leads to considerable physical obscurity, for example, in the quantization of the Maxwellian electromagnetic field.[20, 21] The traditional theory abandons the longitudinal and timelike field polarizations as being "unphysical," and in so doing inevitably loses manifest covariance. Another traditional difficulty[20] is that the little group of the Poincaré group[20] for the massless photon becomes the Euclidean E(2), which is physically obscure. The Lie algebra for the Maxwellian electromagnetic field on the other hand is that of the Lorentz group. These difficulties are accepted because it is traditionally thought that special relativity implies zero photon mass, and that gauge invariance of the first and second kind can be interpreted only in terms of zero photon rest mass. In this paper it is shown that both of these traditional viewpoints are flawed, and that in consequence, the Einstein-de Broglie theory of light is consistent with both special relativity and gauge transformation. We recall for reference that the massless electromagnetic field is summarized in the d'Alembert equation:

$$\Box A_\mu = 0 \tag{4}$$

Quantization[20] of Eq. (1) is straightforward, but that of Eq. (4) is beset with considerable difficulty. From quantization of Eq. (1), for the massive boson, the conclusion is reached that the massive boson is a particle (the photon) with finite mass and three physically meaningful spacelike polarizations, (1), (2), and (3). Quantization[20] of Eq. (4) traditionally proceeds in the Coulomb or Lorentz gauge. To quote from Ryder,[20] "Quantisation of the electromagnetic field suffers from difficulties posed by gauge invariance. The quantisation procedure is outlined in both the radiation (Coulomb) gauge, in which there appear only the two physical (transverse) polarisation states, and in the Lorentz gauge, in which all four polarisation states appear, the formalism being Lorentz covariant. The resulting difficulties are resolved by the method of Gupta and Bleuler." The reader is referred to Ryder[20] for an excellent account of these difficulties. The Coulomb gauge is inconsistent, furthermore, with a nonzero photon rest mass, so that, conversely, finite m_0 implies immediately that the notion of there being only two physically meaningful photon polarization states must be abandoned. One is led ineluctably to the conclusion that there are four physically meaningful photon polarizations ((0) to (3)).

Lorentz gauge quantization[20] in the limit $m_0 \to 0$ is possible only with the Gupta-Bleuler condition,[22] which leads to the conclusion that admixtures of timelike and longitudinal spacelike photon polarizations are physical states.[20] In a diametrically self-contradictory procedure, the traditional theory abandons these physical states as unphysical.

This procedure is logically untenable, and recently[23–28] this has become clear through the discovery of a simple relation between longitudinal and transverse solutions of Maxwell's equations in vacuo:

$$\mathbf{B}^{(3)} = \frac{\mathbf{E}^{(1)} \times \mathbf{E}^{(2)}}{E_0 c \mathrm{i}} = \frac{\mathbf{B}^{(1)} \times \mathbf{B}^{(2)}}{B_0 \mathrm{i}} = B_0 \mathbf{k} \tag{5}$$

Equation (5) comes directly from the original Maxwell equations, without the introduction of scalar and vector potentials, and is an entirely novel relation between physically meaningful electric and magnetic components of the electromagnetic field in vacuo. It can be derived without reference to gauge theory, but is consistent with gauge invariance. Here $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$ are the oscillating transverse components of the electric field, taken to be a plane wave in vacuo. The vector product in Eq. (5) is defined by the Stokes parameter S_3:

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = -S_3 \mathbf{k} \tag{6}$$

In a light beam in which there is some degree of circular polarization, therefore, S_3 is always nonzero, implying that the longitudinal magnetic field $\mathbf{B}^{(3)}$ is nonzero in vacuo. The transverse components in Eq. (5) are the usual vacuum plane wave solutions of Maxwell's equations:

$$\begin{aligned} \mathbf{E}^{(1)} &= \frac{E_0}{\sqrt{2}}(\mathbf{i} - \mathrm{i}\mathbf{j})e^{\mathrm{i}\phi} \qquad & \mathbf{E}^{(2)} &= \frac{E_0}{\sqrt{2}}(\mathbf{i} + \mathrm{i}\mathbf{j})e^{-\mathrm{i}\phi} \\ \mathbf{B}^{(1)} &= \frac{B_0}{\sqrt{2}}(\mathrm{i}\mathbf{i} + \mathbf{j})e^{\mathrm{i}\phi} \qquad & \mathbf{B}^{(2)} &= \frac{B_0}{\sqrt{2}}(-\mathrm{i}\mathbf{i} + \mathbf{j})e^{-\mathrm{i}\phi} \end{aligned} \tag{7}$$

where the phase is

$$\phi = \omega t - \boldsymbol{\kappa} \cdot \mathbf{r}$$

stant t, and $\boldsymbol{\kappa}$ is the wave vector

at a point $\mathbf{r}$. It can also be shown[23] that the concommitant longitudinal electric field $\mathbf{E}^{(3)}$ exists in vacuo, and is related to $\mathbf{B}^{(3)}$ by

$$\mathbf{E}^{(3)} \times \mathbf{B}^{(2)} = \mathbf{B}^{(3)} \times \mathbf{E}^{(2)} \tag{8}$$

so that $\mathbf{E}^{(3)}$ is nonzero if $\mathbf{B}^{(3)}$ is nonzero. The imaginary $\mathrm{i}\mathbf{E}^{(3)}$ is expressible as:

$$\mathrm{i}\mathbf{E}^{(3)} \propto \mathrm{i}\mathbf{E}_0 \mathbf{k} \tag{9}$$

It is worth demonstrating explicitly that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are solutions in vacuo of the Maxwell equations, because

$$\begin{aligned} \nabla \times \mathbf{E}^{(3)} &= 0 \qquad & -\frac{\partial \mathbf{B}^{(3)}}{\partial t} &= 0 \\ \nabla \times \mathbf{B}^{(3)} &= 0 \qquad & \frac{1}{c^2}\frac{\partial \mathbf{E}^{(3)}}{\partial t} &= 0 \\ \nabla \cdot \mathbf{E}^{(3)} &= 0 \qquad & \nabla \cdot \mathbf{B}^{(3)} &= 0 \end{aligned} \tag{10}$$

These relations follow from Eqs. (5) and (9); i.e., $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are solenoidal and phase independent.

In this paper we show that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are natural consequences of the Einstein-de Broglie theory of light, and are physically meaningful magnetic and electric fields. Experiments to detect them would support the theory of Einstein and de Broglie. Equations (5) and (9) are therefore relations between longitudinal and transverse field components in the massless limit of the Einstein-de Broglie theory. This conclusion is consistent with the recent development[24] by the present author of manifestly covariant electrodynamics, using electric and magnetic four vectors. This development is equivalent to the Einstein-de Broglie theory in the massless limit ($m_0 \to 0$), and is a direct consequence of the existence of $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ defined by Eqs. (5) and (9), respectively. It is impossible to reconcile the existence of Eqs. (5) and (9) with traditional thinking, in which $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are abandoned as unphysical. Clearly, $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are formed from physical quantities such as the Stokes parameter S_3. In the Einstein-de Broglie theory, on the other hand, $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are physical fields, components of the four vectors E_μ and B_μ in vacuo. A longitudinal solution of Eq. (1) for $\mathbf{B}^{(3)}$ is given in Section **II**, where it is shown that $\mathbf{B}^{(3)}$ is an exponentially decaying function of ξ in the propagation axis Z of the light beam. The divergence of $\mathbf{B}^{(3)}$ is nonzero for finite m_0, and is given by $-\xi B^{(0)}$, a magnetic monopole in vacuo. The numerical value of ξ (10^{-26} m^{-1}) is so small that for all practical purposes, and for laboratory dimensions, $\mathbf{B}^{(3)}$ is a constant magnetic field, independent of distance and time. Section III derives general solutions of Eq. (1) for the transverse and longitudinal fields of the electromagnetic plane wave in vacuo. A discussion follows of the role of $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ in various experimental tests of the Einstein-de Broglie theory of light, taking into account experimental evidence[17] for finite photon mass.

II. LONGITUDINAL SOLUTIONS OF THE EBP EQUATION IN VACUO

In quantum optics interpreted by Einstein and de Broglie[17] light is constituted by real Maxwellian waves which coexist in spacetime with moving particles—photons. In the Copenhagen interpretation of Bohr, Schrödinger, Pauli, Glauber, and others, on the other hand, light is made up of waves of probability, which cannot coexist in spacetime with photons. In the interpretation of the Einstein-de Broglie school, the photon is massive; in that of the Copenhagen school, it is not necessarily so. The basic electrodynamical equations are therefore (1) and (4), respectively. Although it is frequently asserted[1–16, 20, 21] that the photon is massless in its rest frame, there is no supporting experimental evidence. Indeed, it appears to be impossible to test the hypothesis of zero m_0, because it is impossible to test the implication that the range of electromagnetic radiation is infinite. On the other hand, finite m_0 leads[17] to such observable implications as anomalous red shifts, reported on numerous occasions, and tired light phenomena. Einstein,[18] some years after his theory of special relativity (1905), and during his development of general relativity, proposed that the photon's rest mass is finite, i.e., that the mass of the photon is finite in a frame of reference moving at the speed of light. This leads[17–19] to Eq. (1). It is clear therefore that Einstein saw no contradiction with special relativity in his proposal; i.e., Eq. (1) is Lorentz covariant, even though the photon rest mass, m_0, is nonzero. Several conclusions flow immediately from this proposal.

Firstly, the notion that the photon is massless in the frame of the observer (laboratory frame) because it travels at the speed of light is incorrect if the photon rest mass m_0 is nonzero. In the contemporary description[20] of special relativity, the reason for this is that the quantity

$$C = P_\mu P_\mu \tag{11}$$

is the first (or "mass") Casimir invariant of the Poincaré (inhomogeneous Lorentz) group. Here P_μ is the generator of spacetime translations, first

introduced by Wigner in 1939.[29] A spacetime translation is defined by the operation

$$x'_\mu = x_\mu + a_\mu \tag{12}$$

where x_μ is the distance/time four vector of Minkowski spacetime. P_μ does not appear in the homogeneous Lorentz group,[20] i.e., in a group made up only of boost transformations and Lorentz rotations. The quantity m_0^2 (the square of the rest mass) is therefore invariant to Lorentz transformations, i.e., is the same in the rest frame of the photon (which travels at the speed of light) and in the observer frame. The invariant m_0^2 appears in Eq. (1), which is Lorentz covariant, i.e., fully consistent with special relativity. The latter theory does not imply, therefore, that the photon rest mass is zero. It is clear that Einstein himself[17, 18] saw no inconsistency with special relativity in his proposal of finite m_0, and contemporary theory also shows that m_0^2 is an invariant of the Poincaré group. The Einstein-de Broglie theory of light is therefore consistent with special relativity. This means that the rest frame momentum of the photon (a massive boson) is timelike, not lightlike, and that the photon has rest energy m_0c^2, i.e., that the energy of the photon in its own frame of reference, which moves at the speed of light, is m_0c^2, about 10^{-57} J. The spacelike momentum of the photon in its own rest frame is zero, because it does not move relative to this rest frame. In its rest frame, the photon is thus described by a four vector:

$$\begin{aligned} q_\mu &= (0, 0, 0, im_0c) \\ &= \left(0, 0, 0, i\frac{En_0}{c}\right) \end{aligned} \tag{13}$$

in Minkowski spacetime. In the laboratory frame of the observer, however, the photon's momentum is finite, and the vector (13) is transformed into

$$p_\mu = L_{\mu\nu} q_\nu \tag{14}$$

where $L_{\mu\nu}$ denotes a Lorentz transformation[20] which transforms q_ν into p_μ. Clearly, the latter is observed in the laboratory. Wigner[29] showed that this transformation can be described from a knowledge of the rotation group, and that the little group for q_μ is a rotation group.

As discussed by Vigier[17] the consequences are that photons slow down in the laboratory frame of an observer, although the rest frame must move at the speed of light, which is a universal constant of special relativity.

Photons in the frame of the observer behave like relativistic nonzero mass particles, with rest mass $m_0 \doteq 10^{-68}$ kg. The energy momentum four vector in the observer frame is p_μ, with components[17]

$$p_\mu \equiv \left(\mathbf{p}, \mathrm{i}\frac{En}{c}\right)$$

$$En = h\nu = m_0 c^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{15}$$

$$|\mathbf{p}| \doteq \frac{h\nu}{v} \doteq \frac{h\nu}{c}$$

The velocity of the photon in the observer frame is therefore not c, but v, defined from the Guiding Theorem of de Broglie, the basis of wave mechanics:

$$En_0 = h\nu_0 = m_0 c^2 \tag{16}$$

In other words, the energy of the photon in the rest frame is

$$En_0 = m_0 c^2 = h\nu_0 \tag{17}$$

and its energy in the observer frame is

$$\begin{aligned} En &= m_0 c^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \\ &= h\nu \end{aligned} \tag{18}$$

so that there is a change in the frequency of light from one frame to the other. This is the origin of *observed* distance proportional shifts,[17] the "tired light" of Hubble and Tolman. There are photons, therefore, that move at low velocities and contribute to the mass of the universe. Clearly, this is a consequence of the fact that the field has a finite range, of about 10^{26} m, as discussed in the introduction. This conclusion does not contradict the principle of conservation of energy, because in special relativity, the quantity $P_\mu P_\mu$ is invariant to Lorentz transformation. Therefore, special relativity does not imply that the rest mass of the photon is zero, as in the traditional interpretations.[1–16, 20, 21]

Secondly, if m_0 is not zero, the traditional interpretation[20, 21] of gauge transformations must be revised fundamentally, because it leads to the conclusion that the photon rest mass m_0 is zero and therefore contradicts

the Einstein-de Broglie theory and experimental evidence[17] for finite photon mass. Traditional considerations of gauge transformations also lead to the principle of gauge invariance (eicheninvarianz prinzip), which holds if and only if the photon mass is identically zero. For these reasons, we consider carefully the basic Lagrangian formalism of gauge theory, and modify its interpretation to make it consistent with finite m_0. The result of our considerations is Eq. (3a) of the introduction.

Geometrically, a gauge transformation of the first kind[20, 21] is a rotation in the $(1, 2)$ plane of the "vector" field

$$\boldsymbol{\phi} = \phi_1\mathbf{i} + \phi_2\mathbf{j} \tag{19}$$

through an angle Λ. Under such a rotation, Noether's theorem leads to conserved charge Q in a volume V

$$Q = \mathrm{i}\int\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\mathrm{d}V \tag{20}$$

and a conserved current

$$J_\mu = \mathrm{i}\left(\phi^*\frac{\partial\phi}{\partial x_\mu} - \phi\frac{\partial\phi^*}{\partial x_\mu}\right) \tag{21}$$

The existence of Q and J_μ is based on the invariance of action. When the action is real, the Lagrangian is[20]

$$\mathscr{L} = \left(\frac{\partial\phi}{\partial x_\mu}\right)\left(\frac{\partial\phi^*}{\partial x_\mu}\right) - m^2\phi^*\phi \tag{22}$$

where m is a mass associated with the complex field ϕ, defined by

$$\begin{aligned} \phi &= \frac{\phi_1 + \mathrm{i}\phi_2}{\sqrt{2}} \\ \phi^* &= \frac{\phi_1 - \mathrm{i}\phi_2}{\sqrt{2}} \end{aligned} \tag{23}$$

Since Λ is a constant (an angle in $(1, 2)$) the gauge transformation of the first kind, which can be expressed[20] as

$$\phi \to e^{-i\Lambda}\phi \qquad \phi^* \to e^{i\Lambda}\phi \tag{24}$$

is the same at all points in spacetime, so that at an instant t the same rotation occurs for all points in space. This contradicts special relativity[20] whose universal constant is the speed of light, and which implies that action at a distance is impossible. Electrodynamics cannot, therefore, be invariant to a gauge transformation of the first kind. To comply with special relativity, Λ is made an arbitrary function of spacetime:

$$\Lambda \equiv \Lambda(x_\mu) \tag{25}$$

so defining a gauge transformation of the second kind. For $\Lambda \ll 1$, electrodynamics is invariant to the gauge transformation of the second kind:

$$\phi \to \phi - i\Lambda(x_\mu)\phi \tag{26}$$

Condition (25) implies,[20] however, that $\partial\phi/\partial x_\mu$ does not transform in the same way as ϕ, i.e., does not transform covariantly, so that the action is no longer invariant[20, 21]:

$$\delta\mathscr{L} = J_\mu \frac{\partial\Lambda}{\partial x_\mu} \neq 0 \tag{27}$$

with $\mathscr{L}$ defined by Eq. (22). To preserve the invariance of action under (26), the potential four vector A_μ is introduced through

$$\mathscr{L}_1 = -eJ_\mu A_\mu \tag{28}$$

This implies the need for two more conditions[20]:

$$A_\mu \to A_\mu + \frac{1}{e}\frac{\partial\Lambda}{\partial x_\mu} \tag{29}$$

and

$$\mathscr{L}_2 \equiv e^2 A_\mu A_\mu \phi^* \phi \tag{30}$$

Equations (28) to (30) imply[20]

$$\delta\mathscr{L} + \delta\mathscr{L}_1 + \delta\mathscr{L}_2 = 0 \tag{31}$$

In fundamental gauge theory, therefore, A_μ of the conventional d'Alembert equation is introduced to produce Eq. (31) in association with the extra term (30). So far, nothing has been said about the need for zero mass. We note that if $A_\mu A_\mu = 0$, $\mathscr{L}_2$ is automatically zero.

The field A_μ itself makes a contribution to the Lagrangian, implying the need for an additional $\mathscr{L}_3$ to maintain a zero overall action[20]:

$$\mathscr{L}_3 \equiv -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} \tag{32}$$

where

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{33}$$

the four curl of A_μ, is the electromagnetic field four tensor,[20] an invariant under (29). The complete Lagrangian is therefore

$$\mathscr{L}_{\text{tot}} = \mathscr{L} + \mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3 \tag{34}$$

If the mass, m_0, associated with the electromagnetic field is not zero, then the form of the Lagrangian is changed from (32) to

$$\mathscr{L}_4 = -\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}m_0^2 A_\mu A_\mu \tag{35}$$

and this is invariant to Eq. (29) if and only if

$$m_0^2 A_\mu A_\mu = 0 \tag{36}$$

If $m_0 \neq 0$, then

$$A_\mu A_\mu = 0 \qquad A_\mu \neq 0 \tag{37}$$

is the only alternative possible, as described in the introduction. Conventionally, it is asserted[20, 21] that the invariance of $\mathscr{L}_4$ under (29) means that $m_0 = 0$. However, in the Einstein-de Broglie theory, Eq. (37) is consistent with Eq. (29), and m_0 is quantized as the photon rest mass. Equation (37) is also consistent with Eq. (31) of fundamental gauge theory, because[20] $\mathscr{L}_2 = 0$ if $A_\mu A_\mu = 0$. This implies that

$$\delta \mathscr{L}_2 = 2eA_\mu \left(\frac{\partial \Lambda}{\partial x_\mu} \right) \phi^* \phi = 0 \tag{38}$$

so that

$$\begin{aligned} \delta \mathscr{L} + \delta \mathscr{L}_1 &= -\delta \mathscr{L}_2 \\ &= -2eA_\mu \left(\frac{\partial \Lambda}{\partial x_\mu} \right) \phi^* \phi \\ &= 0 \end{aligned} \tag{39}$$

i.e., the action is conserved as in Eq. (31). The condition (37) for finite m_0 is one in which the EBP equation is invariant to the gauge transformation (29), which is implied by the need to conserve action under the gauge transformation of the second kind, Eq. (26). We therefore conclude that gauge theory does not imply that photon rest mass is zero.

If $m_0 \neq 0$, the quantity $A_\mu A_\mu$ vanishes, implying that $\phi = c|\mathbf{A}|$ where $A_\mu \equiv (\mathbf{A}, i\phi/c)$. This condition is furthermore consistent with Eqs. (1) and (2), which is

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \phi}{\partial t} = 0 \tag{40}$$

that is,

$$\nabla \cdot \mathbf{A} = -\frac{1}{c} \frac{\partial A}{\partial t} \tag{41}$$

Additionally, using the Lorentz condition, Eq. (40),

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = 0 \tag{42a}$$

$$\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = 0 \tag{42b}$$

whose solutions are the Liénard-Wiechert potentials.[1–16, 20] Clearly, Eqs. (42a) and (42b) become the same if $\phi = c|\mathbf{A}|$ in S.I. units.

Fundamental gauge theory does not imply that the photon rest mass must be zero, contrary to much of the current literature.[1–16, 20, 21] Secondly, special relativity, as pointed out by Einstein[17, 18] also does not imply zero photon rest mass. Thirdly, there is experimental evidence,[17] for nonzero m_0, and none for zero photon rest mass. Fourthly, the transverse, radiation, or Coulomb gauge[1–16, 20] is inconsistent with $\phi = c|\mathbf{A}|$, because in that gauge $\phi = 0$, $\mathbf{A} \neq 0$. The Lorentz gauge and Dirac gauge[17] are, on the other hand, consistent with $m_0 \neq 0$.

Having argued in some detail in this way, it becomes easy to see that much of the obscurity in the current thought on electromagnetism is due to the notion that the photon is massless and travels at the speed of light. Both statements contradict experimental evidence.[17] These notions result in "too much gauge freedom," in that Eqs. (26) and (29) can be satisfied with $m_0 = 0$ by the Coulomb, Lorentz, and other gauges. For $m_0 \neq 0$, as in the Einstein-de Broglie theory, the Coulomb gauge is invalidated, but the Lorentz gauge is a direct mathematical consequence of the EBP equation (1). The excess gauge freedom for $m_0 = 0$ results in severe[20, 21] difficulties of quantization of the electromagnetic field, whereas quantization of the EBP equation (a wave equation) is straightforward,[20] leading to longitudinal, physically meaningful, spacelike photon polarization, as well as the two transverse spacelike polarizations. It is natural to expect that a particle, the photon, should have three spacelike polarizations in three physical dimensions, X, Y, and Z.

The $m_0 = 0$ assertion is conventionally associated with the notion that the electromagnetic field is a massless gauge field with two independent components, customarily identified with left and right circular polarization. However, even in the limit $m_0 = 0$, the same Maxwellian field is covariantly described by the four components of A_μ. The Bohm-Aharonov effect[20] shows that A_μ is physically meaningful. Recent work,[23–28] leading to Eq. (5), shows conclusively that there is a well-defined relation between the transverse ((1) and (2)) and longitudinal ((3)) components of solutions of Maxwell's field equations in vacuo. It is straightforward to show that the three magnetic field components form a classical cyclic permutation in the circular basis, (1), (2), and (3): with $B^{(0)} = B_0$

$$\mathbf{B}^{(1)} \times \mathbf{B}^{(2)} = iB^{(0)}\mathbf{B}^{(3)*} \qquad \mathbf{B}^{(3)*} = \mathbf{B}^{(3)} \tag{43a}$$

$$\mathbf{B}^{(2)} \times \mathbf{B}^{(3)} = iB^{(0)}\mathbf{B}^{(1)*} \qquad \mathbf{B}^{(1)*} = \mathbf{B}^{(2)} \tag{43b}$$

$$\mathbf{B}^{(3)} \times \mathbf{B}^{(1)} = iB^{(0)}\mathbf{B}^{(2)*} \qquad \mathbf{B}^{(2)*} = \mathbf{B}^{(1)} \tag{43c}$$

Furthermore, there exist classical permutations involving $\mathbf{E}^{(3)}$. If we assert $\mathbf{E}^{(3)} \equiv E^{(0)}\mathbf{k}$, these are, algebraically,

$$\mathbf{E}^{(1)} \times \mathbf{E}^{(2)} = \mathrm{i}\mathbf{E}^{(0)} c \mathbf{B}^{(3)*} \tag{44}$$

$$\begin{aligned}
\mathbf{E}^{(2)} \times \mathbf{E}^{(3)} &= -\mathbf{E}^{(0)} c \mathbf{B}^{(1)*} \\
\mathbf{E}^{(3)} \times \mathbf{E}^{(1)} &= \mathbf{E}^{(0)} c \mathbf{B}^{(2)*} \\
\mathbf{E}^{(1)} \times \mathbf{B}^{(2)} &= \mathbf{B}^{(0)} \mathbf{E}^{(3)*} \\
\mathbf{E}^{(2)} \times \mathbf{B}^{(3)} &= \mathrm{i}\mathbf{B}^{(0)} \mathbf{E}^{(1)*} \\
\mathbf{E}^{(3)} \times \mathbf{B}^{(1)} &= -\mathbf{B}^{(0)} \mathbf{E}^{(2)*}
\end{aligned} \tag{45}$$

$$\begin{aligned}
\mathbf{E}^{(1)} \times \mathbf{B}^{(1)} &= 0 \\
\mathbf{E}^{(2)} \times \mathbf{B}^{(2)} &= 0 \\
\mathbf{E}^{(3)} \times \mathbf{B}^{(3)} &= 0
\end{aligned} \tag{46}$$

and are reminiscent of the Lie algebra of the Lorentz group,[20] a classical commutator algebra that is built up with boost and rotation generators defined in Minkowski spacetime. However, all the eqns. (45) violate $\hat{T}$ symmetry, which is a consequence of the fact that $\mathbf{E}^{(3)}$ is imaginary and cannot be derived from transverse solutions of Maxwell's equations. Eqns. (45) are not valid equations of electrodynamics while eqn. (44) is valid and identical with eqn. (43a). This does not mean that $\mathbf{E}^{(3)}$ itself violates $\hat{T}$ symmetry.

Before proceeding to the derivation of $\mathbf{B}^{(3)}$ for nonzero m_0, the purpose of this section, it is demonstrated that Lie algebra also applies to the electric and magnetic components of electromagnetic radiation in vacuo (the Maxwellian field) provided that these components are defined as classical field operators directly proportional respectively to the boost and rotation generators of the Lorentz transformation. This is a mathematical demonstration of the fact that if the longitudinal spacelike components of these fields are unphysical (i.e., zero), then the Lie algebraic structure of the Lorentz group is contradicted. This means that the Lorentz transformation itself is incorrectly defined, in that the longitudinal (Z) boost and rotation generator components are incorrectly asserted to be zero. This is equivalent to destroying the geometrical structure of Minkowski spacetime. Even in the Maxwellian limit $m_0 \to 0$, therefore, the assertion that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ are zero results in a mathematical reductio ad absurdum.

That the Maxwell equations in vacuo are the Lorentz covariant equations[20, 21]:

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = 0 \qquad \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\mu} = 0 \tag{47}$$

where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$, the electromagnetic field four tensor. The latter is antisymmetric under Lorentz transformation and its structure can be displayed as

$$F_{\mu\nu} \equiv \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -cB_3 & cB_2 \\ E_2 & cB_3 & 0 & -cB_1 \\ E_3 & -cB_2 & cB_1 & 0 \end{bmatrix} \tag{48}$$

We note that this structure is identical with that of the Lie algebra of the Lorentz group, defined[20] by the dimensionless, boost generator $\hat{K}_i$, an operator, and the rotation generator $\hat{J}_i$, also a dimensionless operator. The Lie algebra of the Lorentz group can be displayed as

$$\hat{J}_{\mu\nu} = \begin{bmatrix} 0 & \hat{K}_1 & \hat{K}_2 & \hat{K}_3 \\ -\hat{K}_1 & 0 & \hat{J}_3 & -\hat{J}_2 \\ -\hat{K}_2 & -\hat{J}_3 & 0 & \hat{J}_1 \\ -\hat{K}_3 & \hat{J}_2 & -\hat{J}_1 & 0 \end{bmatrix} \tag{49}$$

i.e., as

$$\hat{J}_{\mu\nu}(\mu, \nu = 0, \ldots, 3) \begin{bmatrix} \hat{J}_{ij} = -\hat{J}_{ji} = \varepsilon_{ijk}, \hat{J}_k \\ \hat{J}_{i0} = -\hat{J}_{0i} = -\hat{K}_i \end{bmatrix} \tag{50}$$

$$(i, j, k = 1, 2, 3)$$

Equations (49) and (50) are condensed representations of the classical commutator (Lie) algebra of the Lorentz group[20]:

$$\left[\hat{J}_X, \hat{J}_Y\right] = \mathrm{i}\hat{J}_Z \text{ and cyclic permutations} \tag{51a}$$

$$\left[\hat{K}_X, \hat{K}_Y\right] = -\mathrm{i}\hat{J}_Z \text{ and cyclic permutations} \tag{51b}$$

$$\left[\hat{K}_X, \hat{J}_Y\right] = \mathrm{i}\hat{K}_Z \text{ and cyclic permutations} \tag{51c}$$

$$\left[\hat{K}_X, \hat{J}_X\right] = 0 \text{ etc.} \tag{51d}$$

The geometrical equivalence of (48) and (49) means that

$$\begin{aligned}
\hat{E}_i &= E_0 \hat{K}_i \\
\hat{B}_i &= B_0 \hat{J}_i
\end{aligned} \tag{52}$$

where $\hat{E}_i$ and $\hat{B}_i$ are classical electric and magnetic field operators, a result that is implied by the proportionality of the classical operator matrices $\tilde{F}_{\mu\nu}$ and $\hat{J}_{\mu\nu}$. In the Cartesian basis (X, Y, Z),

$$\left[\hat{B}_X, \hat{B}_Y\right] = \mathrm{i}B_0\hat{B}_Z \text{ and cyclic permutations} \tag{53a}$$

$$\left[\hat{E}_X, \hat{E}_Y\right] = -\mathrm{i}cB_0\hat{B}_Z \text{ and cyclic permutations} \tag{53b}$$

$$\left[\hat{E}_X, \hat{B}_Y\right] = \mathrm{i}B_0\hat{E}_Z \text{ and cyclic permutations} \tag{53c}$$

$$\left[\hat{E}_X, \hat{B}_X\right] = 0 \text{ etc.} \tag{53d}$$

Equations (53) represent a classical operator equivalent of the vector products in Eqs. (36), where the Maxwellian fields are vectors in space, and not operators defined in spacetime.

The ansatz (52) is based on the fundamental Lie algebraic structure of Minkowski spacetime, and implies the following:

1. The classical electric field operator $\hat{E}_i$ is proportional to a boost generator, and the classical magnetic field operator $\hat{B}_i$ is proportional to a rotation generator in Minkowski spacetime.
2. If the longitudinal component operators $\hat{B}_Z$ and $\hat{E}_Z$ are asserted to be zero, or unphysical, the structure of the Lie algebra is destroyed in Eqs. (53). For example, if $\hat{B}_Z = \hat{E}_Z = \hat{0}$, $[\hat{B}_X, \hat{B}_Y] \doteq \hat{0}$ and from the structure of $\hat{J}_i$ in Eq. (52), this is mathematically incorrect. Explicitly,

$$\left[\hat{B}_X, \hat{B}_Y\right] = B_0^2\left[\hat{J}_X, \hat{J}_Y\right] = B_0^2\hat{J}_Z \neq 0 \tag{54}$$

because[20]

$$\hat{J}_X \equiv -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{55a}$$

$$\hat{J}_Y \equiv -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{55b}$$

$$\hat{J}_Z \equiv -\mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{55c}$$

3. The Maxwell equations (47) are seen to be relations between boost and rotation generators defined in spacetime:

$$\frac{\partial J_{\mu\nu}}{\partial x_\mu} = 0 \qquad \frac{\partial \tilde{J}_{\mu\nu}}{\partial x_\mu} = 0 \tag{56}$$

and are thus given a precise geometrical interpretation. In this light, it is seen that the d'Alembert equation (4) is also geometrical in nature:

$$\Box\, \hat{L}_\mu = 0 \tag{57}$$

where $\hat{J}_{\mu\nu}$ is the four curl of $\hat{L}_\mu$:

$$\hat{J}_{\mu\nu} = \frac{\partial \hat{L}_\nu}{\partial x_\mu} - \frac{\partial \hat{L}_\mu}{\partial x_\nu} \tag{58}$$

4. It may be seen precisely that the conventional notion that the Maxwellian $\hat{B}_Z$ (and $\hat{E}_Z$) is unphysical is equivalent to the geometrically incorrect assertion

$$\hat{J}_Z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{59}$$

which by implication habitually[1-16, 20, 21] replaces the correct rotation generator (55c).

There is of course no experimental evidence for Eq. (59), even in the massless limit $m_0 \to 0$ conventionally associated with the Maxwellian field.

Our geometrical interpretation of the Maxwell field equations is a direct logical consequence of the geometry of Minkowski spacetime itself and of the theory of special relativity. This is consistent with the fact that Einstein's considerations of the Maxwell equations led to his formulation of special relativity. If it is asserted that longitudinal solutions of Maxwell's equations be unphysical, special relativity is contradicted and the structure of the Lorentz group and its associated Lie algebra is destroyed. There is no experimental evidence whatsoever that the longitudinal solutions of Maxwell's equations in vacuo are unphysical, and there is no evidence for $m_0 = 0$.

The commutator relations (51a) and (53a) lead to a method of quantization of the Maxwellian field simply by noting the ordinary angular momentum commutator relations of quantum mechanics. In Cartesian terms,

$$\left[\hat{J}_X, \hat{J}_Y\right] = i\hbar \hat{J}_Z \tag{60}$$

are structurally identical with Eq. (51a) except for $\hbar$ (which has the units of angular momentum). In quantum mechanics, the $\hat{J}$ operators in Eq. (60) are angular momentum operators. Quantized angular momentum is therefore a consequence of the classical rotation generator,[20] as is well known. The quantized equivalent of Eq. (53a) must therefore be

$$\left[\hat{B}_X, \hat{B}_Y\right] = i\hbar\left(\frac{B_0}{\hbar}\right)\hat{B}_Z \tag{61}$$

to balance units, symmetries, and dimensions on the left and right sides. This implies

$$\hat{B} = B^{(0)}\frac{\hat{J}}{\hbar} \tag{62}$$

which is identical with the result obtained recently by the present author[25] using an independent method of derivation. Therefore,

$$\hat{B}_Z = B^{(0)}\frac{\hat{J}_Z}{\hbar} \tag{63}$$

is the elementary longitudinal component of the quantized Maxwellian magnetic field in vacuo. In the same way that $\hbar$ is the archetypical elementary quantum of angular momentum, $B^{(0)}$ is the elementary quantum of magnetic flux density of the Maxwellian field in vacuo.

The eigenvalues of $\hat{J}_Z$ in Eq. (63) may be identified with those of a massless boson (the "conventional" photon), i.e., $\hbar M_J$, where $M_J = \pm 1$, so that the classical limit of Eq. (63) is

$$\mathbf{B}_Z = B^{(0)}\mathbf{k} \tag{64}$$

which is Eq. (5) in Cartesian terms instead of a circular basis. Equation (5) is therefore geometrically consistent with the Lie algebra of the Lorentz group. The generalization of our development to $m_0 \not\div 0$ is now straightforward.

Having considered in some detail the geometrical structure of the Lorentz group, we revert to a simpler development of the EBP equation (1), solving it as a classical eigenvalue equation with the differential operator:

$$\square \equiv -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{65}$$

The order of magnitude of ξ is such that

$$\square A_\mu \doteq 10^{-52} A_\mu \tag{66}$$

which closely approximates the d'Alembert equation (4). It is clear, therefore, that the classical interpretation of the EBP field closely approximates the Maxwellian field. However, in the EBP field, the Coulomb gauge is inconsistent with Eq. (66), which must be written in terms of the spacelike **A** as

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = \xi^2\mathbf{A} \tag{67}$$

with $\phi = c|\mathbf{A}|$. In the Galilean limit this equation becomes

$$\nabla^2\mathbf{A} = \xi^2\mathbf{A} \tag{68}$$

Using the relation

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{69}$$

it can be seen that the equation

$$\nabla^2\mathbf{B} = \xi^2\mathbf{B} \tag{70}$$

is the same as Eq. (68), because

$$\nabla^2(\nabla \times \mathbf{A}) = \xi^2\nabla \times \mathbf{A} \tag{71a}$$

$$\nabla \times \nabla^2\mathbf{A} = \nabla \times \xi^2\mathbf{A} \tag{71b}$$

In considering the Galilean limit, we have removed the time dependence in the solution for **B** of Eq. (70). Furthermore, since

$$\nabla^2\mathbf{B} = 10^{-52}\mathbf{B} \sim \mathbf{0} \tag{72}$$

describes the magnetic component in vacuo of an electromagnetic field closely resembling the Maxwellian field, we know that the time-independent solution to Eq. (70) must be the longitudinal component, defined in the propagation axis Z. The solution to Eq. (70) in Cartesian terms is therefore

$$\mathbf{B} = B^{(0)} \exp(-\xi Z)\mathbf{k}$$
$$|\mathbf{B}| = B_Z \tag{73}$$

and since $\xi \sim 10^{-26}\ \text{m}^{-1}$, this is for all practical purposes identical with Eq. (64) of the Maxwellian field. Several physical consequences follow from Eqs. (64) and (73):

1. The longitudinal solution for **B** of the EBP field, Eq. (73), is for all practical purposes identical with the corresponding Maxwellian solution, Eq. (64). By the caveat "for all practical purposes" we imply laboratory dimensions and time scales. On a cosmic scale, in which $Z \sim 1/\xi$, Eq. (73) is different from Eq. (64) in general. In the "tired light" terminology of Hubble,[17] **B** becomes a "tired field" if Z is big enough (ca. 10^{26} m).
2. Physically meaningful, practically identical, and longitudinal solutions exist for **B** from the EBP and Maxwell equations, the former being considered as a classical wave equation. To assert $\mathbf{B} = \mathbf{0}$ in Eq. (64) is mathematically incorrect in the Maxwellian field, because it corresponds to the assertion (59) in spacetime. For all practical purposes, therefore, this assertion is incorrect in the EBP field. Quantization of the EBP field[20] confirms this conclusion, leading to a physically meaningful longitudinal photon polarization.
3. Since the EBP and Maxwellian fields are practically (i.e., in the laboratory) identical, the EBP field obeys the various commutator relations of this paper for all practical purposes, and the transverse EBP solutions are practically those of Eqs. (7). In the cosmology of light from distant sources, however, this simple classical interpretation is no longer tenable.

Quantization of the EBP field is straightforward,[20] whereas that of the Maxwellian field is obscure. Although the rest mass m_0 of the photon is very small, it is essential that it be rigorously nonzero to maintain a logical and self-consistent, physically meaningful, structure for the quantized electromagnetic field in vacuo. If this is done, quantization results in a consistent particle interpretation[20] in terms of a massive boson, with eigenvalues $M_J\hbar, M_J = -1, 0, +1$. The three polarization vectors of the

quantized EBP field are orthonormal and spacelike; i.e., there are physically meaningful longitudinal and transverse components. The little group of Wigner[29] is a physically meaningful rotation group, utilizing the three dimensions of space. If $m_0 = 0$, on the other hand, the constraint $A_\mu A_\mu = 0$ is conventionally lost, resulting in "too much gauge freedom." The two Casimir invariants[20] of the Poincaré group vanish for $m_0 = 0$, meaning that physical quantities that are invariant under the most general type of Lorentz transformation must vanish identically for the massless gauge field. This implies $A_\mu A_\mu = 0$, if $A_\mu A_\mu$ is to be an invariant of the Poincaré group, diametrically contradicting the conventional use of gauge freedom for a massless particle, i.e., contradicting the conventional assertion that $A_\mu A_\mu \not\equiv 0$ for $m_0 = 0$. Thus, the conventional assertion $A_\mu A_\mu \not\equiv 0$ for $m_0 = 0$ is geometrically unsound, i.e., contradicts the geometry of Minkowski spacetime, a geometry that requires $A_\mu A_\mu$ to be an invariant of the Poincaré (inhomogeneous Lorentz) group. We are forced to conclude that the widespread use of the Coulomb gauge, in which $A_\mu A_\mu = 0$, is relativistically incorrect. The conventional assertion that m_0 must be zero because $A_\mu A_\mu$ is nonzero is also basically incorrect, because $A_\mu A_\mu$ is always zero in vacuo.

It is the habitual use of the Coulomb (or "transverse") gauge that more than any other factor leads to the conventional assertion that the electromagnetic field can have no longitudinal solution that is physically meaningful. The Coulomb gauge is relativistically incorrect, and is inconsistent with finite photon rest mass, for which there is experimental evidence.[17] The widespread use of the Coulomb gauge[1–16, 20, 21] should therefore be viewed with caution. It is obvious that quantization in the Coulomb gauge cannot be consistent with special relativity, because its use is equivalent to the incorrect assertion (59). These difficulties are frequently compounded in the literature by a series of misstatements, traceable to the relativistically incorrect assertion $A_\mu A_\mu \not\equiv 0$. For example, it is frequently asserted that the Lorentz gauge does not define A_μ uniquely. This is true if and only if $A_\mu A_\mu \not\equiv 0$. If $A_\mu A_\mu = 0$, then the Lorentz condition defines A_μ uniquely. Quantization in the Coulomb gauge is therefore a mathematically incorrect procedure, and we discard its results as meaningless. In other words it is meaningless to assert that the Maxwellian field has only two transverse polarizations.

Quantization of the Maxwellian field in the Lorentz gauge[20] retains manifest covariance, but is physically obscure. It also relies on the notion that the gauge field is massless, so that quantization of the field must lead to a massless photon. In consequence, the internally inconsistent notion $A_\mu A_\mu \not\equiv 0$ is habitually retained in the Lorentz gauge. This immediately leads to the difficulty that the Lagrangian has to be modified with a gauge fixing term, a procedure that leads to a non-Maxwellian equation of motion.[20] Even with this artifice, the conjugate momentum field $\mathbf{\Pi}^0$ vanishes,[20] and the traditional method is forced to assert that the Lorentz condition, within whose framework the method is developed, cannot hold as an operator identity. This difficulty is habitually resolved by the method of Gupta and Bleuler,[20] a method that results in the conclusion that admixtures of timelike and longitudinal spacelike photon polarizations are physical states.[20] Despite this conclusion, these states are abandoned as unphysical in order to comply with the results of Coulomb gauge quantization, which, as we have just seen, are incorrect. Quantization of the Maxwellian field, regarded as a massless gauge field, is therefore inconsistent and physically obscure.

In considerations of the Poincaré group, the notion of a massless gauge field, habitually associated with the Maxwellian field, leads to the little group[20, 29] E(2), the Euclidean group of rotations and translations in a plane. The physical significance of this little group is obscure.[20] Its Lie algebra does not correspond to that of a rotation group, but it is the group that is needed to maintain a lightlike vector invariant under the most general Lorentz transformation. This suggests that the notion of a massless field is physically meaningless. The traditional line of reasoning, however, considers a massless particle traveling in the propagation axis (Z) described by a lightlike four vector k_μ. Invariance of k_μ under the most general type of Lorentz transformation leads to the Lie algebra:

$$\begin{aligned} \left[\hat{L}_1, \hat{L}_2\right] &= 0 \\ \left[\hat{J}_3, \hat{J}_1\right] &= \mathrm{i}\hat{L}_3 \\ \left[\hat{L}_2, \hat{J}_3\right] &= \mathrm{i}\hat{L}_1 \end{aligned} \tag{74}$$

where

$$\hat{L}_1 \equiv \hat{K}_1 - \hat{J}_2$$
$$\hat{L}_2 \equiv \hat{K}_2 + \hat{J}_1$$

Thus,

$$\begin{aligned} \left[\hat{L}_1, \hat{L}_2\right] &= \left[\hat{K}_1, \hat{K}_2\right] + \left[\hat{K}_1, \hat{J}_1\right] \\ &\quad - \left[\hat{J}_2, \hat{K}_2\right] - \left[\hat{J}_2, \hat{J}_1\right] \\ &= 0 \end{aligned} \tag{75}$$

In Cartesian terms, $X = 1$, $Y = 2$, $Z = 3$ and if we attempt to apply to Eq. (75) the Lorentz group algebra of Eqs. (51),

$$\left[\hat{K}_1, \hat{K}_2\right] = \left[\hat{K}_X, \hat{K}_Y\right] = -\mathrm{i}J_Z \tag{76a}$$

$$\left[\hat{J}_2, \hat{J}_1\right] = -\left[\hat{J}_X, \hat{J}_Y\right] = -\mathrm{i}\hat{J}_Z \tag{76b}$$

$$\left[\hat{K}_1, \hat{J}_1\right] = \left[\hat{K}_X, \hat{J}_X\right] = 0 \tag{76c}$$

$$\left[\hat{J}_2, \hat{K}_2\right] = \left[\hat{J}_Y, \hat{K}_Y\right] = 0 \tag{76d}$$

we obtain

$$\left[\hat{L}_1, \hat{L}_2\right] = 2\mathrm{i}\hat{J}_Z \neq \hat{0} \tag{77}$$

Since in the Lorentz group

$$\hat{J}_Z \neq \hat{0} \tag{78}$$

in general, Eq. (77) contradicts Eq. (75).

Therefore, the most general Lorentz transformation that leaves the lightlike momentum vector $\mathbf{k}_\mu$ invariant cannot be described by the Lie algebra of the Lorentz group. This implies that the notion of lightlike momentum (a massless particle traveling at the speed of light), is not relativistically self-consistent. This is another way of demonstrating that the quantization of a massless field into a massless particle is beset with obscurity; i.e., we are led to the conclusion that the Maxwellian field has no meaning in quantum theory. Attempts to impose a meaning lead into physical obscurity as we have described. In the Einstein-de Broglie theory of light, the quantization of the EBP field leads directly and without difficulty[20] to a particle interpretation of light in terms of a massive boson. Quantum/classical equivalence in the EBP field is therefore clear. The only physically meaningful and consistent interpretation is to accept the photon as a massive boson whose classical field is described by the classical limit of the EBP equation. The mathematical limit of this field for zero mass is the Maxwellian field. Direct quantization of the Maxwellian field, regarded as a classical massless gauge field, is physically obscure. The quantized Maxwellian field must therefore be defined as being for all practical purposes the quantized EBP field, with which it is practically identical because photon mass is numerically very small.

III. TRANSVERSE SOLUTIONS IN VACUO FOR FINITE PHOTON MASS

The EBP equation can be written in terms of the tensor $F_{\mu\nu}$ defined in Eq. (43) as

$$\frac{\partial F_{\mu\nu}}{\partial x_\mu} = -\xi^2 A_\nu \sim 0 \tag{79}$$

and, as we have seen, the Lie algebra associated with $F_{\mu\nu}$ is given by Eqs. (43)–(46). Therefore, transverse solutions of the EBP equation in its classical limit obey the classical cross products in Eqs. (43)–(46). Using Eq. (43a) with the longitudinal solution of the EBP equation (73), we obtain

$$\mathbf{B}^{(1)} = \frac{B_0}{\sqrt{2}}(\mathrm{i}\mathbf{i} + \mathbf{j})e^{\mathrm{i}\phi}\,e^{-\xi Z/2} \qquad \mathbf{B}^{(2)} = \mathbf{B}^{(1)*} \tag{80}$$

and its complex conjugate. The difference between this solution and the equivalent Eq. (7c) for the Maxwell equations can be expressed by replacing the wave vector of the Maxwell equations by

$$\kappa_Z \to \kappa_Z - \frac{\mathrm{i}\xi}{2} \text{ for polarization 1}$$

$$\kappa_Z \to \kappa_Z + \frac{\mathrm{i}\xi}{2} \text{ for polarization 2}$$

At visible frequencies, the order of magnitude of the Maxwellian κ in vacuo is given by

$$\kappa_Z = \frac{\omega}{c} \sim \frac{10^{15}}{10^8} \sim 10^7 \text{ m}^{-1} \tag{81}$$

so that at these frequencies κ_Z is about 33 orders of magnitude greater than ξ. For all practical purposes, therefore, the transverse solutions of the classical limit of the EBP equation are identical with those of the Maxwell equations.

This is a simple demonstration in the classical limit that the fields associated with the EBP and Maxwell equations contain physically meaningful longitudinal as well as transverse components in vacuo. In the next section we discuss several experimental consequences of physically meaningful longitudinal fields when electromagnetic radiation interacts with matter. Firstly, however, we review the available experimental evidence for finite photon mass, following a recent account by Vigier.[17]

IV. DISCUSSION

There is available an increasing amount of evidence for finite photon rest mass, upon which is based the theory of Einstein and de Broglie. A recent experiment by Mizobuchi and Ohtake[17] has demonstrated for single photons the simultaneity of classical wave and particle behavior in light. This has demonstrated for the first time that the Copenhagen interpretation cannot be valid, but supports the Einstein-de Broglie interpretation as reviewed recently by Vigier,[17] an interpretation that implies, for example, that photons are emitted from a source in quanta of energy with well-defined directionality. The wave associated with a single photon has a physical reality. Light is constituted by massive bosons (photons) controlled or piloted[17] by real surrounding spin one fields. The motion of the photon is thus controlled by a quantum potential. The photons are the only directly observable elements of light and behave in Minkowski space-time as relativistic particles with finite mass. Light is also constituted in the Einstein-de Broglie theory by physically meaningful fields (waves), which, as we have seen, obey Maxwell's equations for all practical purposes, essentially because the photon rest mass is finite (10^{-68} kg) but small. These fields are described by complex vector waves, which also describe photon motion. Thus, if there is a longitudinal photon polarization, there must be a longitudinal field polarization, as described already. Longitudinal field solutions of the EBP equation were first derived by Schrödinger and de Broglie and, in general, the EBP equation has longitudinal and transverse WAVE solutions.[17] Since these are also wave solutions of Maxwell's equations for all practical purposes, it becomes clear that Maxwell's equations must have physically meaningful longitudinal solutions. The relation of these to the transverse solutions has only recently become clear,[23–28] as described in Sections II and III of this paper.

Following Vigier's recent description[17] there are several consequences of finite photon mass. The r dependence of the Coulomb potential is replaced by that of the Yukawa potential:

$$V_Y \propto \frac{\exp(-\xi Z)}{Z} \tag{82}$$

There exist low velocity photons (i.e., photons traveling at considerably less than the speed of light c), whose small but finite mass contributes to that of the universe. There is, thirdly, a red shift proportional to $\exp(-\xi Z)$, which can be applied to explain recent astronomical observations of anomalous red shifts from several distant sources, such as quasars. These "tired light" phenomena originate in the EBP equation and may account for observed anomalies in double-star motions, galaxy clusters, observed variations of the Hubble "constant," and other evidence reviewed in the literature.[17] A photon with finite rest mass behaves relativistically in the frame of observation, leading to the expectation[17] of a direction dependent anisotropy in the frequency of light in the observer frame. Such an anisotropy has been observed experimentally by Hall et al.[30] in the direction of the apex of the 2.7 K background of microwave radiation. These Boulder experiments are currently being repeated in Copenhagen by Poulsen and coworkers.[17] Experimental evidence for the Einstein-de Broglie theory of light has also been reviewed by Vigier[17] in the following areas:

1. Super-luminal action at a distance, a facet of Einstein's interpretation of light
2. The question of locality or nonlocality of the quantum potential
3. Direct experimental testing of Heisenberg's uncertainty principle using single photons
4. Experimental testing for the existence of particle trajectories in light (einweg/welcherweg)
5. Testing the existence of physically meaningful waves without the presence of particles, for example, the recent experimental observation by Bartlett and Corle[31] of the Maxwell displacement current in vacuo
6. Testing directly the existence of the quantum potential with intersecting laser beams and laser-induced fringe patterns

There is, therefore, a considerable amount of experimentation in progress concerning the existence of finite photon mass, and it is no longer tenable to assert[1–16, 20, 21] that the photon mass is zero.

Similarly, it is not reasonable to assert that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ must be zero, "irrelevant," "unphysical," or similar, as in much of the contemporary literature. It is in fact implied, but not specifically stated, in the work of de Broglie and Schrödinger[17] that $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ must exist. They exist, as we have seen, both for finite photon mass and in the Maxwellian limit, but finite photon rest mass is essential for a natural quantization of the electromagnetic field. For all intents and purposes, therefore, evidence for $\mathbf{B}^{(3)}$ and $\mathbf{E}^{(3)}$ is evidence for finite photon mass, and corroboration for

other sources of evidence quoted already. The present author has proposed a number of different magneto-optic experiments[23–28] that would test for $\mathbf{B}^{(3)}$ through its interaction with matter, using its characteristic square root dependence on light intensity I_0 (W m^{-2}). In free space, fundamental electrodynamics leads to[23–28]

$$|\mathbf{B}^{(3)}| \sim 10^{-7} I_0^{1/2} \tag{83}$$

and assuming that $\mathbf{B}^{(3)}$ acts as a magnetic field whose time average is nonzero, it is to be expected[23–28] that there exist the following effects (collected details in Ref. 26) proportional to the square root of laser intensity, provided that the laser is circularly polarized: (1) inverse Faraday effect (magnetization due to $\mathbf{B}^{(3)}$), (2) optical Faraday effect (azimuth rotation due to $\mathbf{B}^{(3)}$), (3) effects of $\mathbf{B}^{(3)}$ in NMR (preliminary observations reported in Ref. 32) and ESR spectroscopy, (4) Cotton-Mouton effect due to $\mathbf{B}^{(3)}$, (5) forward–backward birefringence due to $\mathbf{B}^{(3)}$, and (6) reinterpretation of antisymmetric light scattering and similar phenomena in terms of $\mathbf{B}^{(3)}$.

Finally, we propose the Bohm-Aharonov effect due to $\mathbf{B}^{(3)}$ of a circularly polarized laser, which replaces the solenoid, or iron whisker[20] of the conventional Bohm–Aharonov effect. The Bohm-Aharonov effect[20] indicates that the vector potential in quantum mechanics is physically meaningful, and that the vacuum has a nontrivial topology. It is therefore one of the most incisive effects in contemporary electrodynamics. The experiment has been repeated independently several times and consists of placing a small solenoid between two slits, which are used to generate interference fringes due to electron beams. The magnetic flux density **B** (tesla) is confined within the solenoid, and is inaccessible to the interfering electrons passing through the two slits. Despite this, the solenoid is observed experimentally[20] to produce a shift in the interference pattern (or fringes) set up by the electrons. This shift is due to the curl of the vector potential **A** set up outside the solenoid. Essentially, **A** changes the electron wave function

$$\psi = |\psi| \exp\left(i \frac{\mathbf{p} \cdot \mathbf{r}}{\hbar} \right) \tag{84}$$

because **p**, the electron momentum, is changed to $\mathbf{p} - e\mathbf{A}$, where e is the electronic charge. This does not occur in classical mechanics, but in quantum theory, the electronic wave function, and thus the electron, is influenced by **A** even though it travels in regions where magnetic flux density **B** is zero. This means that there is *nonlocality* in the integral

$\oint \mathbf{A} \cdot d\mathbf{r}$ (Ref. 20). The Bohm-Aharonov effect is therefore evidence for this type of nonlocality.

The shift is given in meters by

$$\Delta x = \frac{L\lambda}{d} \frac{e}{h} \Phi \tag{85}$$

where λ is the wavelength of the electron beam entering the two slits, L is the distance between the screen containing the two slits and the detector plane, d is the distance between the two slits, and

$$\Phi = \int \mathbf{B} \cdot d\mathbf{S} = \oint \mathbf{A} \cdot d\mathbf{r} \tag{86}$$

is a surface integral.

It is clear that if the solenoid is replaced by a thin, circularly polarized, laser beam, there should be a Bohm-Aharonov effect due to $\mathbf{B}^{(3)}$ in which this field shifts the interference pattern of the electrons, with $\mathbf{B}$ of Eq. (85) replaced by $\mathbf{B}^{(3)}$. This shift should be proportional to the square root of the laser intensity, reverse with the sense of circular polarization of the laser (because $\mathbf{B}^{(3)}$ changes sign), and disappear if the laser is linearly polarized or incoherently polarized. This laser-induced fringe displacement would be a particularly interesting investigation of the nature of $\mathbf{B}^{(3)}$, and of its concommitant $\mathbf{A}^{(3)}$. Presumably $\mathbf{B}^{(3)}$ is confined to the radius of the laser beam, and $\mathbf{A}^{(3)}$ exists outside this beam, as in a solenoid generating a conventional, longitudinal, magnetostatic field. The experiment would prove both the existence and the nonlocality of $\mathbf{A}^{(3)}$.

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